Exploring the Photonic Spectrum of Magneto-optical Superlattice with Magnetic Fields

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Abstract: In this study, we investigate how an external magnetic field that can have any direction affects a superlattice’s photonic band structure, composed of alternating dielectric and plasma layers. By deriving the dispersion equations, we show that the photonic spectrum of the superlattice loses its degeneracy and splits into two branches due to the external magnetic field. Moreover, our findings reveal a direction-independent transition from a photonic insulator to a photonic conductor. Interestingly, our results indicate that a superlattice that was previously completely photo-isolating can become entirely photo-conducting, regardless of the direction of the external magnetic field.

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References and links
1. Introduction

Because of the high mobility of the free charge carriers in conducting media, their electrodynamic parameters strongly depend on external magnetic fields. Owing to this, various characteristics of optical devices containing conducting constituents can be altered with the use of an external magnetic field [1–11]. Among various optical systems, optical superlattices, regular structures of alternating layers with different optical parameters, attract great attention. Due to this, it is of interest to analyze how the external magnetic field can alter the photonic characteristics of a superlattice consisting of alternating dielectric and conducting layers. Usually, the external magnetic field is assumed to have some constant direction, and the effect of the magnitude of the external magnetic field on the photonic properties of the system is investigated. In the present study, we examine how the external magnetic field’s direction can change the superlattice’s photonic properties. A distinctive feature of the system under consideration is its anisotropy since the conducting layers of the superlattice become electrodynamically anisotropic in the presence of the external magnetic field. It is well known how to obtain the photonic spectrum of the isotropic superlattices, which are described in terms of $2 \times 2$ unit-cell transfer matrix, see [12]. As for the anisotropic ones, the formalism of the $2 \times 2$ transfer matrix becomes inapplicable and has to be considerably modified. Because of the anisotropy, the dimensions of the unit-cell transfer matrix become $4 \times 4$, that considerably complicates the theoretical analysis of the problem [13]. For instance, to describe the eigenstates of the electromagnetic field in the superlattice, one must determine the eigenvalues of the unit-cell transfer matrix. To do this, one has to find all roots of an algebraic equation of the fourth degree. In the present study, we show that such a problem is actually reduced to the solution of a quadratic equation, whose roots are described by well-known formulas.

Another critical question is how many new branches in the photonic spectrum of the superlattice can be generated by the external magnetic field. On the one hand, as well known, in the absence of the external magnetic field, there is one dispersion relation, see [12], for degenerated Bloch electromagnetic eigenmode corresponding to propagation of the electromagnetic wave along the axis of the superlattice. On the other hand, the number of eigenstates of the electromagnetic field is associated with the number of the eigenvalues of the unit-cell transfer matrix, whose value for the $4 \times 4$ unit-cell transfer matrix is equal to four. Although the unit-cell transfer matrix generally has four different eigenvalues, it turns out that the photonic spectrum of the superlattice in the presence of the external magnetic field is described by two dispersion equations for two different Bloch phases.

It should be noted that there are a few numerical methods that can be used to simulate the photonic properties of optical systems, see, for instance, [14, 15]. It is essential to underline that such numerical algorithms can be applied quite effectively to isotropic optical structures. However, their applicability to anisotropic ones is minimal. They have to be modified in a general case where tensors with all non-zero components describe the anisotropy, as it takes place in the problem under consideration. The analytical results which we obtain in the present study significantly simplify the problem of describing photonic properties of the magneto-optical superlattices because we can readily find frequency bands where the superlattice operates as a
conductor of electromagnetic waves and where it does as an insulator.

It is worth mentioning that in paper [16] we studied impact of the external magnetic field on propagation of the electromagnetic waves in general case where both the direction of the wave propagation and the orientation of the magnetic field are arbitrary. To do this, we numerically analyzed the eigenstates of the electromagnetic field within the framework of the transfer matrix method. In the present study we use the purely analytical approach to derive the dispersion relations, which are of great interest in the theory of propagation of electromagnetic waves in magneto-optical layered media.

2. Problem formulation

We examine the photonic spectrum of electromagnetic waves propagating along the axis of a superlattice, which is an array composed of identical unit cells, in the presence of static and homogenous magnetic field $\mathbf{H}_0$ of arbitrary direction, as shown in Fig. (1). Without loss of generality, we consider that the direction of the magnetic field $\mathbf{H}_0$ lies in the $x – y$ plane. The unit cell contains two layers: the dielectric $a$-slab and the magneto-optical plasma-containing $b$-slab with the constant thicknesses $d_a$ and $d_b$, respectively. Below we assume that the magneto-optical $b$-slab is semiconductor and the superlattice is the sequence of the alternating layers of the isolator and semiconductor. The width of each unit $(a, b)$ cell $d = d_a + d_b$ is the spatial period of the superlattice. The dielectric $a$-slabs are specified by permittivity $\varepsilon_a$, refractive index $n_a = \sqrt{\varepsilon_a}$, wave number $k_a = n_a k_0$, and phase shift $\varphi_a = k_a d_a$. Quantity $k_0$ is $k_0 = \omega / c$, where $\omega$ is the frequency of the electromagnetic wave.

![Fig. 1: A sketch of the structure. Quantity $\theta$ is the angle between external magnetic field $\mathbf{H}_0$ and the $x$-axis; the direction of the external magnetic field lies in the $x – y$ plane.](image)

In the presence of the external magnetic field $\mathbf{H}_0$, the magneto-optical $b$-slabs are specified by the conductivity tensor $\hat{\sigma}$. When the direction of the external magnetic field $\mathbf{H}_0$ lies in the $x – y$
plane, the conductivity tensor \( \hat{\sigma} \) of the magneto-optical \( b \)-layers has the following elements: [16],

\[
\sigma_{xx} = \frac{\omega_p^2}{4\pi} \frac{\omega_x^2 + (\nu - i\omega)^2}{\omega_x^2 + (\nu - i\omega)^2} (\nu - i\omega), \quad (1)
\]

\[
\sigma_{yy} = \frac{\omega_p^2}{4\pi} \frac{\omega_y^2 + (\nu - i\omega)^2}{\omega_y^2 + (\nu - i\omega)^2} (\nu - i\omega), \quad (2)
\]

\[
\sigma_{zz} = \frac{\omega_p^2}{4\pi} \frac{\nu - i\omega}{\omega_c^2 + (\nu - i\omega)^2}, \quad (3)
\]

\[
\sigma_{xy} = \sigma_{yx} = \frac{\omega_p^2}{4\pi} \frac{\omega_x\omega_y}{\omega_c^2 + (\nu - i\omega)^2} (\nu - i\omega), \quad (4)
\]

\[
\sigma_{xz} = -\sigma_{zx} = \frac{\omega_p^2}{4\pi} \frac{\omega_x}{\omega_c^2 + (\nu - i\omega)^2} (\nu - i\omega), \quad (5)
\]

\[
\sigma_{yz} = -\sigma_{zy} = \frac{\omega_p^2}{4\pi} \frac{\omega_x}{\omega_c^2 + (\nu - i\omega)^2}, \quad (6)
\]

where \( \omega_p = \sqrt{4\pi e^2 n / m} \) is the plasma frequency, \( n \) is the concentration of the charge carriers in the plasma, \( m \) is the effective mass of the charge carrier with charge \(-e\), \( \omega_c = |e|H_0/mc \) is the cyclotron frequency of the charge carrier in the external magnetic field \( H_0 \); quantities \( \omega_x \) and \( \omega_y \) are \( \omega_x = eH_{0x}/mc \) and \( \omega_y = eH_{0y}/mc \). Expressions in Eqs. (1)-(6) are obtained within the Drude model.

The electromagnetic field in the superlattice is a superposition of the static external magnetic field \( H_0 \), the alternating electric \( E(x, y, z, t) \) field, and the alternating magnetic \( H(x, y, z, t) \) field. Directing the \( x \)-axis normally to the layers of the superlattice, the alternating electromagnetic field reads as,

\[
E(x, y, z, t) = E(x) \exp(-i\omega t), \quad H(x, y, z, t) = H(x) \exp(-i\omega t). \quad (7)
\]

As it follows from the Maxwell equations,

\[
\text{curl}E = ik_0H, \quad \text{curl}H = -ik_0\varepsilon_aE, \quad (8)
\]

inside the dielectric \( a \)-layer, and inside the magneto-optical \( b \)-slab

\[
\text{curl}E = ik_0H, \quad \text{curl}H = -ik_0\varepsilon_bE. \quad (9)
\]

According to the conductivity tensor \( \hat{\sigma} \), we have introduced the permittivity tensor \( \hat{\varepsilon} \) of the magneto-optical \( b \)-layer in the presence of the external magnetic field,

\[
\varepsilon_{\alpha\beta} = \varepsilon_L \delta_{\alpha\beta} + \frac{4i\pi}{\omega} \sigma_{\alpha\beta}, \quad \alpha, \beta = x, y, z. \quad (10)
\]

Here quantity \( \varepsilon_L \) is the permittivity of the lattice of the magneto-optical layer.

### 3. Distribution of electromagnetic field

The dielectric layer is isotropic and the form of the electromagnetic field inside it can be easily established. Thus, inside the dielectric \( a \)-layer of the \( n \)th unit cell, the \( x \)-dependent part of the
The quantities $b$ and $E$ are the magneto-optical electromagnetic field inside the magneto-optical superposition of four modes with four different wave vectors. Thus, inside the magneto-optical electromagnetic field inside the dielectric layer of the left-hand side and right-hand side of the magneto-optical layer. Such a system can be written as one matrix equation for two-component quantity $y(x)$

$$y(x) = 2 \text{matrix} \times 2 \text{matrix} \times \text{index}$$

or $E_n(x) = \text{exp}\{ik_n(x - x_{an})\}, \quad H_n(x) = \text{exp}\{ik_n(x - x_{an})\}, \quad E_n(x) = [nB_n(\pm) \text{exp}\{ik_n(x - x_{an})\} - B_n(\pm) \text{exp}\{ik_n(x - x_{an})\}], \quad H_n(x) = -A_n(\pm) \text{exp}\{ik_n(x - x_{an})\} + A_n(\pm) \text{exp}\{ik_n(x - x_{an})\}, \quad A_n(\pm) \text{are the four complex amplitudes of the electromagnetic field inside the dielectric layer of the nth unit cell.}

The $x$-dependent part of the electromagnetic field inside the magneto-optical $b$-slab is a superposition of four modes with four different wave vectors. Thus, inside the magneto-optical $b$-slab of the nth unit cell the components of the electric and magnetic fields are

$$E_\alpha(n)(x) = \sum_{\beta = 1}^{4} E_{\alpha\beta} C_\beta(n) \text{exp}\{ik_\beta(x - x_{b_\beta})\}, \quad H_\alpha(n)(x) = \sum_{\beta = 1}^{4} H_{\alpha\beta} C_\beta(n) \text{exp}\{ik_\beta(x - x_{b_\beta})\}, \quad \text{where } x_{b_\beta} \leq x \leq x_{b_{\beta+1}}, \text{ and index } \alpha \text{ can be } x, y, \text{ or } z; x_{b_\beta} \text{ and } x_{b_{\beta+1}} \text{ are, respectively, the positions of the left-hand side and right-hand side of the magneto-optical } b\text{-layer within the nth unit cell.}

The quantities $C_1(n), C_2(n), C_3(n),$ and $C_4(n)$ in Eqs. (17) and (18) are the complex amplitudes of the electromagnetic field inside the magneto-optical $b$-layer of the nth unit cell; the quantities $k_1, k_2, k_3,$ and $k_4$ are the $x$-components of the wave vectors of the four electromagnetic modes inside the magneto-optical $b$-layer.

Let us consider the equations for the electric and magnetic fields within the magneto-optical $b$-slab. From Eqs. (9) we obtain that

$$\text{curl}\text{curl}E = k_0^2 \hat{\epsilon} E, \quad H = (1/ik_0)\text{curl}E.$$

The first equation in Eq. (19) gives three scalar equations,

$$E_x''(x) = \left[\epsilon_{xx} E_x(x) + \epsilon_{xy} E_y(x) + \epsilon_{xz} E_z(x)\right]/\epsilon_{xx}, \quad E_y''(x) + k_0^2 \epsilon_{xy} E_x(x) + k_0^2 \epsilon_{yx} E_y(x) + k_0^2 \epsilon_{xz} E_z(x) = 0, \quad E_z''(x) + k_0^2 \epsilon_{xz} E_x(x) + k_0^2 \epsilon_{zx} E_y(x) + k_0^2 \epsilon_{zz} E_z(x) = 0.$$

Here the prime means the derivative with respect to $x$. After substitution of Eq. (20) into Eq. (21) and Eq. (22), we eliminate quantity $E_x(x)$ and get a system of two equations for $E_y(x)$ and $E_z(x)$. Such a system can be written as one matrix equation for two-component quantity $E_z(x) = (E_y(x), E_z(x))^T$,

$$E_z''(x) + k_0^2 \hat{\epsilon} E_z(x) = 0, \quad \text{where elements of } 2\times2 \text{ matrix } \hat{\epsilon} \text{ are related to the components of the permittivity tensor of the magneto-optical } b\text{-layer, which is given by Eq. (10)},$$

$$\epsilon_{11} = \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}/\epsilon_{xx}, \quad \epsilon_{12} = \epsilon_{yz} - \epsilon_{xz} \epsilon_{yx}/\epsilon_{xx},$$

$$\epsilon_{21} = \epsilon_{zy} - \epsilon_{xy} \epsilon_{zx}/\epsilon_{xx}, \quad \epsilon_{22} = \epsilon_{zz} - \epsilon_{xz} \epsilon_{zx}/\epsilon_{xx}.$$
It can be checked that \( \varepsilon_{11} = -\varepsilon_{12} \).

The matrix equation in Eq. (23) has solutions that depend on \( x \) as \( E_{1}(x) \propto \exp(ikx) \), where \( k \) is the \( x \)-component of the wave vector within the magneto-optical \( b \)-layer. One can see that a general solution of Eq. (23) can be formulated in terms of matrix \( \hat{M}(k) \),

\[
\hat{M}(k) = k_0^2 \hat{I} - k^2 \hat{I},
\]

where \( \hat{I} \) is the identity matrix. Thus, all possible values of \( k \) — \( k_1, k_2, k_3, \) and \( k_4 \), can be determined from the condition \( \det \hat{M}(k) = 0 \) that leads to a biquadratic equation

\[
k^4 - (\varepsilon_{11} + \varepsilon_{22}) k_0^2 + \left( \varepsilon_{11} \varepsilon_{22} + \varepsilon_{12}^2 \right) k_0^4 = 0. \tag{26}
\]

From Eq. (26) we obtain the four possible components of the wave vector,

\[
k_{1,2} = \mp n_{b}^{-}(k_0, \quad k_{3,4} = \mp n_{b}^{+}(k_0, \tag{27}
\]

where the two refractive indexes \( n_{b}^{-} \) and \( n_{b}^{+} \) of the magneto-optical \( b \)-layer are

\[
n_{b}^{-} = \sqrt{\frac{1}{2} \left[ \varepsilon_{11} + \varepsilon_{22} - \sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 - 4 \varepsilon_{12}^2} \right]}, \tag{28}
\]

\[
n_{b}^{+} = \sqrt{\frac{1}{2} \left[ \varepsilon_{11} + \varepsilon_{22} + \sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 - 4 \varepsilon_{12}^2} \right]}. \tag{29}
\]

The quantities \( E_{sx} \) and \( E_{sz} \), where \( s = 1, 2, 3, 4 \), in the electric field components \( E_{r}(x) \) and \( E_{z}(x) \) are the nontrivial solutions of the homogeneous system of equations

\[
\begin{pmatrix}
M_{11}(k_s) & M_{12}(k_s) \\
M_{21}(k_s) & M_{22}(k_s)
\end{pmatrix}
\begin{pmatrix}
E_{ys} \\
E_{cz}
\end{pmatrix}
= 0, \quad s = 1, 2, 3, 4. \tag{30}
\]

They are easily found. Using them inline with Eq. (20) and Eq. (17) we also find the coefficients \( E_{sx} \), so that for quantities \( E_{sx}, E_{sy}, \) and \( E_{sz} \), where \( s = 1, 2, 3, 4 \), we have

\[
E_{sx} = \frac{1}{E_{sx}} \left| \frac{M_{11}(k_s)}{M_{12}(k_s)} \right| E_{sz} - \varepsilon_{sy}, \quad E_{sy} = 1, \quad E_{cz} = -\frac{M_{11}(k_s)}{M_{12}(k_s)}, \quad s = 1, 2, 3, 4. \tag{31}
\]

Using the second equation in Eq. (19) and Eq. (17) one finds that coefficient \( \mathcal{H}_{sx} \) of the magnetic field is zero, whereas the remaining ones \( \mathcal{H}_{sy} \) and \( \mathcal{H}_{sz} \) are expressed through quantities \( E_{sy} \) and \( E_{sz} \), respectively,

\[
\mathcal{H}_{sx} = 0, \quad \mathcal{H}_{sy} = -(k_s/k_0)E_{sz}, \quad \mathcal{H}_{sz} = (k_s/k_0)E_{sy}, \quad s = 1, 2, 3, 4. \tag{32}
\]

We have obtained the general expressions which completely define the distribution of the electromagnetic field within the dielectric \( a \)- and magneto-optical \( b \)-layers. Inside the dielectric layer the expressions for the components of the electromagnetic field contain the four complex amplitudes: \( A_{n}^{(-)}, A_{n}^{(+)} \), \( B_{n}^{(-)} \), and \( B_{n}^{(+)} \). Within the magneto-optical layer the expressions for the components of the electromagnetic field contain the four complex amplitudes as well: \( C_{1}^{(n)} \), \( C_{2}^{(n)} \), \( C_{3}^{(n)} \), and \( C_{4}^{(n)} \). With the use of the expressions describing the electromagnetic field inside the dielectric \( a \)- and magneto-optical \( b \)-layers, we can obtain the unit-cell transfer matrix, which describes the wave transmission through the unit cell of the superlattice. Since inside the layers of the superlattice the electromagnetic field is defined by the four amplitudes, the unit-cell transfer matrix has dimensions \( 4 \times 4 \).
4. Transfer matrix

In the superlattice composed of the $a$- and $b$-layers, the complex amplitudes corresponding to the $a$-layer and those corresponding to the $b$-layer are not independent: they are related to each other by the continuous boundary conditions for the tangential ($y$- and $z$-) components of the electromagnetic field. Using the boundary conditions at the left ($x = x_{b_n}$) and right ($x = x_{a_{n+1}}$) boundaries of the magneto-optical $b$-layer within the $n$th unit cell, we get two matrix relations for quantities $F_n$, $F_{n+1}$, and $C$,

\[
F_n = \left( A_n^{(+)} , A_n^{(-)} , B_n^{(+)} , B_n^{(-)} \right)^T ,
\]

\[
F_{n+1} = \left( A_{n+1}^{(+)} , A_{n+1}^{(-)} , B_{n+1}^{(+)} , B_{n+1}^{(-)} \right)^T ,
\]

\[
C = \left( C_n^{(1)} , C_n^{(2)} , C_n^{(3)} , C_n^{(4)} \right)^T .
\]

Quantities $F_n$ and $F_{n+1}$ define the state of the electromagnetic field within the dielectric layers of the $n$th and $(n+1)$th unit cells, respectively; quantity $C$ defines the state of the electromagnetic field within the magneto-optical $b$-layer of the $n$th unit cell.

One can obtain that the conditions expressing the continuity of the tangential components of the electromagnetic field at the left ($x = x_{b_n}$) and right ($x = x_{a_{n+1}}$) boundaries of the magneto-optical $b$-layer within the $n$th unit cell are:

\[
\hat{T}_a \hat{D}_a F_n = \hat{T}_b C , \quad \hat{T}_b \hat{D}_b C = \hat{T}_a F_{n+1} .
\]

Matrices $\hat{T}_a$, $\hat{T}_b$, $\hat{D}_a$, and $\hat{D}_b$ are given by

\[
\hat{T}_a = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & -n_a & n_a \\
n_a & -n_a & 0 & 0
\end{pmatrix}, \quad
\hat{T}_b = \begin{pmatrix}
\mathcal{E}_{y_1} & \mathcal{E}_{y_2} & \mathcal{E}_{y_3} & \mathcal{E}_{y_4} \\
\mathcal{E}_{z_1} & \mathcal{E}_{z_2} & \mathcal{E}_{z_3} & \mathcal{E}_{z_4} \\
\mathcal{H}_{y_1} & \mathcal{H}_{y_2} & \mathcal{H}_{y_3} & \mathcal{H}_{y_4} \\
\mathcal{H}_{z_1} & \mathcal{H}_{z_2} & \mathcal{H}_{z_3} & \mathcal{H}_{z_4}
\end{pmatrix},
\]

\[
\hat{D}_a = \begin{pmatrix}
\exp(i \varphi_a) & 0 & 0 & 0 \\
0 & \exp(-i \varphi_a) & 0 & 0 \\
0 & 0 & \exp(i \varphi_a) & 0 \\
0 & 0 & 0 & \exp(-i \varphi_a)
\end{pmatrix},
\]

\[
\hat{D}_b = \begin{pmatrix}
\exp(-i \varphi_b^{(-)}) & 0 & 0 & 0 \\
0 & \exp(i \varphi_b^{(-)}) & 0 & 0 \\
0 & 0 & \exp(-i \varphi_b^{(+)}) & 0 \\
0 & 0 & 0 & \exp(i \varphi_b^{(+)})
\end{pmatrix} .
\]

Here quantities $\varphi_b^{(+)}$ and $\varphi_b^{(-)}$ are

\[
\varphi_b^{(\pm)} = k^{(\pm)} b , \quad k^{(\pm)} = n_{b}^{(\pm)} k_0 .
\]

From Eq. (36) we find the following recurrent relation for the complex amplitudes of the electromagnetic field inside the dielectric $a$-layers of the $(n+1)$th and $n$th unit cells,

\[
F_{n+1} = \hat{Q} F_n , \quad \hat{Q} = \hat{T}_a^{-1} \hat{T}_b \hat{D}_b \hat{D}_a \hat{T}_a .
\]

Matrix $\hat{Q}$ has the meaning of the unit-cell transfer matrix since it describes the wave transmission through the entire $(a,b)$ unit cell.
Therefore, if for the independent eigenvalues we choose

$$\lambda$$

to obtain the dispersion relations describing the eigenstates of the electromagnetic field, we have
to find the eigenvalues of the unit-cell transfer matrix \( \hat{Q} \). They can be determined from algebraic
equation \( \det(\hat{Q} - \lambda I) = 0 \) for unknown quantity \( \lambda \), where \( I \) is the identical matrix. Since the
dimensions of the unit-cell transfer matrix \( \hat{Q} \) are \( 4 \times 4 \), it has the four eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \) and
\( \lambda_4 \) satisfying a polynomial equation of the fourth degree,

$$\det(\hat{Q} - \lambda I) = 0 \to \lambda^4 + u\lambda^3 + r\lambda^2 + u\lambda + 1 = 0,$$

(42)

Coefficients \( u \) and \( r \) are expressed through the parameters of the problem as follows,

$$u = \left[ M_\gamma \sin\left(\varphi_b^{(-)}\right) + M_\sigma \sin\left(\varphi_b^{(+)}\right) \right] \sin(\varphi_a)$$

$$-2 \cos(\varphi_a) \left[ \cos\left(\varphi_b^{(-)}\right) + \cos\left(\varphi_b^{(+)}\right) \right],$$

(43)

$$r = 2 + 2 \cos\left(\varphi_b^{(-)}\right) \cos\left(\varphi_b^{(+)}\right) [1 + \cos(2\varphi_a)]$$

$$- \left[ M_\gamma \sin\left(\varphi_b^{(-)}\right) \cos\left(\varphi_b^{(+)}\right) + M_\sigma \cos\left(\varphi_b^{(-)}\right) \sin\left(\varphi_b^{(+)}\right) \right] \sin(2\varphi_a)$$

$$+ M_\sigma M_\gamma \sin\left(\varphi_b^{(-)}\right) \sin\left(\varphi_b^{(+)}\right) \sin^2(\varphi_a).$$

(44)

Here quantities \( M_\gamma \) are given by

$$M_\gamma = \frac{n_a}{n_b^{(-)}} + \frac{n_b^{(-)}}{n_a}, \quad M_\sigma \equiv \frac{n_a}{n_b^{(+)}} + \frac{n_b^{(+)}}{n_a}. \quad (45)$$

Due to the symmetry of the coefficients in quartic equation in Eq. (42), it can be reduced to a
quadratic equation \( w^2 + uw + r - 2 = 0 \) by means of a change of variables \( \lambda + \lambda^{-1} = w \). As a
result, the roots \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \) of Eq. (42) are

$$\lambda_{1,2} = \frac{1}{4} \left[-u - \sqrt{D} \pm \sqrt{(-u - \sqrt{D})^2 - 16} \right],$$

(46)

$$\lambda_{3,4} = \frac{1}{4} \left[-u + \sqrt{D} \pm \sqrt{(-u + \sqrt{D})^2 - 16} \right],$$

(47)

$$D = u^2 - 4r + 8.$$  

(48)

From Eqs. (46) and (47) one can see that only two of four eigenvalues are independent, since

$$\lambda_1\lambda_2 = 1, \quad \lambda_3\lambda_4 = 1. \quad (49)$$

Therefore, if for the independent eigenvalues we choose \( \lambda_1 \) and \( \lambda_3 \), then the remaining two ones
\( \lambda_2 \) and \( \lambda_4 \) are expressed through \( \lambda_1 \) and \( \lambda_3 \), respectively,

$$\lambda_2 = 1/\lambda_1, \quad \lambda_4 = 1/\lambda_3.$$  

(50)

Owing to Eq. (50) we can express the four eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \) of the unit-cell
transfer matrix through two independent quantities \( \gamma_+ \) and \( \gamma_- \),

$$\lambda_1 = \exp(i\gamma_+), \quad \lambda_2 = \exp(-i\gamma_+), \quad \lambda_3 = \exp(i\gamma_-), \quad \lambda_4 = \exp(-i\gamma_-).$$  

(51-52)
Here \( \gamma_+ \) and \( \gamma_- \) are the Bloch phases of the electromagnetic eigenmodes of the superlattice in the presence of the external magnetic field. From these equations one can see that the two Bloch phases \( \gamma_+ \) and \( \gamma_- \) satisfy the relations
\[
\cos(\gamma_+) = (\lambda_1 + \lambda_2)/2, \quad \cos(\gamma_-) = (\lambda_3 + \lambda_4)/2, \quad (53)
\]
or the same
\[
\cos(\gamma_+) = \left( -u - \sqrt{D} \right)/4, \quad \cos(\gamma_-) = \left( -u + \sqrt{D} \right)/4. \quad (54)
\]
After substitution of Eqs. (43), (44), and (48) into Eqs. (54), one finds the equations for the Bloch phases \( \gamma_+ \) and \( \gamma_- \). Such a form of the equations is not final and they can be simplified: after some algebra one can obtain that quantity \( D = u^2 - 4r + 8 \) can be represented as a perfect square,
\[
D = \left[ 2 \left( \cos(\varphi_b^{(-)}) - \cos(\varphi_b^{(+)}) \right) \cos(\varphi_a) + M_+ \sin(\varphi_b^{(+)}) \right. \\
\left. - M_- \sin(\varphi_b^{(-)}) \right] \sin(\varphi_a)^2. \quad (55)
\]
This allows us to get rid of the irrationality \( \sqrt{D} \) in Eqs. (54): one can use the expression in the curly brackets in Eq. (55) as \( \sqrt{D} \). Then substituting it inline with Eq. (43) into Eqs. (54), we obtain the final form of the desired dispersion equations,
\[
\cos(\gamma_-) = \cos(\varphi_a) \cos(\varphi_b^{(-)}) - \frac{1}{2} M_- \sin(\varphi_a) \sin(\varphi_b^{(-)}), \quad (56)
\]
\[
\cos(\gamma_+) = \cos(\varphi_a) \cos(\varphi_b^{(+)}) - \frac{1}{2} M_+ \sin(\varphi_a) \sin(\varphi_b^{(+)}) \quad (57)
\]
Here we have obtained an important result: in the presence of the external magnetic field the photonic spectrum of the superlattice is characterized by two dispersion relations for two Bloch modes \( \gamma_- \) and \( \gamma_+ \). This is the principal dissimilarity from the case of the isotropic superlattice, where only one Bloch mode can exist.

It is not difficult to check that in the limit of the vanishing external magnetic field \( H_0 \to 0 \), the Bloch phases \( \gamma_- \) and \( \gamma_+ \) go to the Bloch phase \( \gamma \) of the isotropic superlattice,
\[
\gamma_-(\omega) \to \gamma(\omega), \quad \gamma_+(\omega) \to \gamma(\omega) \quad \text{as} \quad H_0 \to 0. \quad (58)
\]
Indeed, in the absence of the external magnetic field the refractive indexes \( n_b^{(-)} \) and \( n_b^{(+)} \) of the magneto-optical \( b \)-layer become equal to each other,
\[
n_b^{(-)} = n_b^{(+)} = n_b = \sqrt{\varepsilon_b} \quad \text{when} \quad H_0 = 0, \quad (59)
\]
where
\[
\varepsilon_b = \varepsilon_L - \omega_p^2/\omega(\omega + iv) \quad (60)
\]
is the Drude permittivity of the magneto-optical \( b \)-layer. Due to this
\[
M_- = M_+ = n_a/n_b + n_b/n_a \quad \text{and} \quad \varphi_b^{(-)} = \varphi_b^{(+)} = \varphi_b = n_b k_0 d_b \quad \text{when} \quad H_0 = 0. \quad (61)
\]
Consequently, as it follows from Eqs. (56) and (57), when the external magnetic field decreases, the photonic spectrums \( \gamma_- = \gamma_-(\omega) \) and \( \gamma_+ = \gamma_+(\omega) \) merge into one \( \gamma = \gamma(\omega) \) which satisfies the dispersion equation,
\[
\cos(\gamma) = \cos(\varphi_a) \cos(\varphi_b) - \frac{1}{2} \left( n_a/n_b + n_b/n_a \right) \sin(\varphi_a) \sin(\varphi_b), \quad H_0 = 0 \quad (62)
\]
where $\varphi_a = n_a k_0 d_a$ and $\varphi_b = n_b k_0 d_b$. This is well known dispersion equation [12] describing propagation of the electromagnetic waves in the superlattice composed of the \textit{isotropic} $a$- and $b$-slabs with refractive indexes $n_a = \sqrt{\varepsilon_a}$ and $n_b = \sqrt{\varepsilon_b}$, respectively.

6. Results

Now we apply the analytical results obtained in the preceding sections to a superlattice consisting of the dielectric and semiconductor slabs. We consider that the dielectric $a$- and semiconductor $b$-layers are, respectively, quartz with refractive index $n_a = 2$ and indium antimonide with the concentration of the charge carriers $n = 10^{15}$ cm$^{-3}$ and permittivity of the lattice $\varepsilon_L = 17.8$. For simplicity, we assume that there is no absorption in the layers of the superlattice, so that the relaxation frequency $\nu$ in the plasma of $b$-layers is zero:

$$\nu = 0.$$  \hspace{1cm} (63)

The thicknesses of the dielectric $a$- and semiconductor $b$-layers are $d_a = 20 \times \delta$ and $d_b = 0.5 \times \delta$, where $\delta = c/\omega_p$ and $\omega_p = \sqrt{4\pi ne^2/m}$.

Figure (2) displays the photonic spectrum of the superlattice in the absence of the external magnetic field, $H_0 = 0$. where $\omega/\omega_p = 0.116$ THz, where, in the absence of the external magnetic field, $H_0 = 0$, the superlattice cannot conduct light since there are no propagating states of the electromagnetic field within region $0 \leq \omega \leq 0.05 \times \omega_p$, see Fig. (2). We show how the photonic band structure of the superlattice changes in the presence

![Fig. 2: Photonic spectrum of the superlattice in the absence of the external magnetic field, $H_0 = 0$.](image-url)
Fig. 3: Photonic band structure of the superlattice for two values of the strength $H_0$ of the external magnetic field: $H_0 = 0.1$ T (figure a) and $H_0 = 0.4$ T (figure b).

of the external magnetic field of magnitudes $H_0 = 0.1$ T and $H_0 = 0.4$ T, within the frequency range $0 \leq \omega \leq 0.05 \times \omega_p$ and for the angles of the external magnetic field from $0^\circ \leq \theta \leq 180^\circ$. Angle $\theta$ defines the direction of the external magnetic field $\mathbf{H}_0$ relatively to the $x$-axis; the direction of the external magnetic field $\mathbf{H}_0$ lies in the $x-y$ plane.

In the absence of the absorption, the two refractive indexes $n^{(\pm)}_0$ of the $b$-layer are either pure imaginary or pure real. Therefore, the right hand sides of the dispersion equations in Eqs. (56) and (57) are real. Therefore, within some frequency bands where

$$|\cos(\gamma_-)| \leq 1 \quad \text{or} \quad |\cos(\gamma_+)| \leq 1$$

there are pure real solutions for the Bloch phase $\gamma_-$ or for the Bloch phase $\gamma_+$. It is apparent that such solutions with real $\gamma_-$ or $\gamma_+$ define the photonic pass bands where the propagating states of the electromagnetic field exist.

Figure (3a) shows the photonic band structure when the strength of the external magnetic field is 0.1 T. The white area corresponds to the points $(\theta, \omega)$ where there are no propagating states of the electromagnetic field. At such values of angle $\theta$ and frequency $\omega$ the superlattice cannot conduct light. The blue area displays points $(\theta, \omega)$ where the propagating states of the electromagnetic field exist, i.e. where the superlattice can conduct light. It is remarkable that the external magnetic field results in the existence of the propagating states of the electromagnetic field within the photonic stop band of the superlattice in the absence of the external magnetic field. A distinctive feature of the band structure shown in Fig. (3a) is that the size of the photonic pass band strongly depends on the direction $\theta$ of the external magnetic field. Indeed, as one can see, the size of the lower photonic pass band varies from approximately $0.02 \times \omega_p$ at $\theta = 0^\circ$ to zero at $\theta = 90^\circ$. The size of the next photonic pass bands strongly depends on the angle $\theta$ too.

Figure (3b) displays the photonic band structure when the strength of the external magnetic field is 0.4 T. Unlike the previous figure, one can see the region from frequency which approximately equals $0.04 \times \omega_p$ (the black horizontal dashed line) to frequency $0.05 \times \omega_p$, which is completely filled with the propagating states of the electromagnetic field, independently on the direction $\theta$ of the external magnetic field. Therefore, at any orientation $\theta$ of the external magnetic field the superlattice can conduct light.

7. Conclusion

In summary, we have examined the impact of the direction of the external magnetic field on the photonic spectrum of the superlattice composed of the alternating dielectric and magneto-optical
plasma-containing layers. We have successfully solved this problem using analytical methods, obtaining dispersion equations that describe the propagation of electromagnetic waves in the superlattice in the case of the arbitrary direction of the external magnetic field. Our results show that the photonic spectrum of the superlattice changes significantly when the external magnetic field is applied, giving rise to two distinct branches in the photonic spectrum of the superlattice. Interestingly, the magnetic field induces propagation states within the photonic stop band of the superlattice, where the propagating modes of the electromagnetic field are typically forbidden. Consequently, there is a transition from a photo-isolating state to a photo-conducting one, which depends on the strength and direction of the external magnetic field. Moreover, the size of the photonic pass band is found to be heavily influenced by the direction of the external magnetic field. However, we have also discovered that at specific magnetic field strengths, the superlattice undergoes a complete transition from a photo-isolating to a photo-conducting state, where the propagating states of the electromagnetic field are observed inside the photonic stop band, regardless of the direction of the external magnetic field.