Singular value representation of the coherence Poincaré sphere

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Abstract

The so-called coherence Poincaré sphere was recently introduced for geometrical visualization of the state of two-point spatial coherence of a random electromagnetic beam. The formalism and its interpretation strongly utilized a specific decomposition of the Gram matrix of the cross-spectral density (CSD) matrix. In this work, we show that the interpretation of the coherence Poincaré sphere is obtained exclusively and straightforwardly via the singular value decomposition of the CSD matrix.

1 Introduction

Optical coherence theory deals with the analysis and consequences of randomness in optical fields which gives rise to partial (spatial and temporal) coherence of light [1, 2]. A topic of substantial interest in the last two decades has been the coherence properties of vectorial light [3], and among the very recent results is the geometrical representation called the coherence Poincaré sphere [4, 5]. This formalism displays the spatial coherence of a random electromagnetic beam similarly as the traditional Poincaré sphere [6, 7] depicts the beam's polarization characteristics, and it is the first graphical representation of electromagnetic two-point coherence. Illustrative geometrical interpretations of this kind are often found to be extremely useful in physics as evidenced by the conventional Poincaré sphere in polarization optics and the Bloch sphere in quantum mechanics [8]. In optics, the Poincaré sphere and its variants have found important applications, e.g., in the context of full Poincaré beams [9], orbital angular momentum [10], higher-order polarization states [11, 12], vector fields [13], and scalar two-beam interference [14].

In the previous works [4, 5], the derivation and interpretation of the coherence Poincaré sphere was based on decomposing the Gram matrix of the cross-spectral density (CSD) into two parts in full analogy to the division of the polarization matrix [1] into parts corresponding to a completely unpolarized and fully polarized beams. The present work complements and extends these earlier studies by employing the singular value decomposition (SVD) of the CSD matrix, which was utilized in [5] and more extensively studied in [15]. More precisely, we derive the formalism of the coherence Poincaré sphere using the SVD exclusively, and show that this approach leads to a physical interpretation for the sphere as a geometric representation of the intertwined coherence and polarization information conveyed by the singular values and vectors of the CSD matrix.

2 Discussion

Consider a random, polychromatic, and statistically stationary electromagnetic beam field. The spatial coherence properties of the field at two positions $\mathbf{r}_1$ and $\mathbf{r}_2$ on a transversal plane with respect to the propagation direction are described in the space-frequency domain by the CSD matrix [1, 2, 6, 16]

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$$\mathbf{W}\left(\mathbf{r}_1,\mathbf{r}_2,\omega\right) = \langle \mathbf{E}^{*}(\mathbf{r}_1,\omega)\mathbf{E}^{\mathrm{T}}(\mathbf{r}_2,\omega)\rangle, \tag{1}$$

taken as an average over an ensemble of monochromatic (transverse, two component) electric field realizations $\mathbf{E}(\mathbf{r},\omega)$ at angular frequency ω. The angle brackets, asterisk, and superscript T stand for the ensemble average, complex conjugate, and matrix transpose, respectively. The SVD of the CSD matrix is written as [5, 6, 7]

$$\mathbf{W}_{12} = \mathbf{U}\mathbf{D}\mathbf{V}^{\dagger}, \tag{2}$$

where the dagger denotes Hermitian conjugation and $\mathbf{W}_{12} = \mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega)$. From now on we do not explicitly show the frequency dependence of various quantities. In Eq. (2), $\mathbf{U} = [\hat{\mathbf{u}}_{+},\hat{\mathbf{u}}_{-}]$ and $\mathbf{V} = [\hat{\mathbf{v}}_{+},\hat{\mathbf{v}}_{-}]$ are unitary matrices, and $\mathbf{D} = \mathrm{diag}[\nu_{+},\nu_{-}]$, with ν_{+} and ν_{-} representing the singular values of the CSD. The complex unit vectors $\hat{\mathbf{u}}_{\pm}$ and $\hat{\mathbf{v}}_{\pm}$ in the columns of $\mathbf{U}$ and $\mathbf{V}$ are the left and right singular vectors of the CSD matrix obeying

$$\mathbf{W}_{12}\hat{\mathbf{v}}_{\pm} = \nu_{\pm}\hat{\mathbf{u}}_{\pm}, \tag{3}$$
$$\mathbf{W}_{12}^{\dagger}\hat{\mathbf{u}}_{\pm} = \nu_{\pm}\hat{\mathbf{v}}_{\pm}. \tag{4}$$

The singular values are real and satisfy $\nu_{+} \geq \nu_{-} \geq 0$. Furthermore, their squares coincide with the eigenvalues of the Gram matrices of the CSD and its Hermitian adjoint given, respectively, by [4, 5]

$$\boldsymbol{\Omega}_{12} = \mathbf{W}_{12}^{\dagger}\mathbf{W}_{12}, \tag{5}$$
$$\boldsymbol{\Omega}_{21} = \mathbf{W}_{12}\mathbf{W}_{12}^{\dagger}, \tag{6}$$

where $\boldsymbol{\Omega}_{12} = \boldsymbol{\Omega}(\mathbf{r}_1,\mathbf{r}_2,\omega)$, $\boldsymbol{\Omega}_{21} = \boldsymbol{\Omega}(\mathbf{r}_2,\mathbf{r}_1,\omega)$, and we employed the quasi-Hermiticity property $\mathbf{W}_{12}^{\dagger} = \mathbf{W}_{21}$. We note that the Gram matrices contain second-order coherence information only. The vectors $\hat{\mathbf{v}}_{\pm}$ and $\hat{\mathbf{u}}_{\pm}$ fulfil the eigenvalue equations

$$\boldsymbol{\Omega}_{12}\hat{\mathbf{v}}_{\pm} = \nu_{\pm}^{2}\hat{\mathbf{v}}_{\pm}, \tag{7}$$
$$\boldsymbol{\Omega}_{21}\hat{\mathbf{u}}_{\pm} = \nu_{\pm}^{2}\hat{\mathbf{u}}_{\pm}. \tag{8}$$

Expressions for $\nu_{\pm}^{2}$ can therefore be obtained from the characteristic equation $\det(\boldsymbol{\Omega}_{12} - \nu_{\pm}^{2}\boldsymbol{\sigma}_0) = 0$, where $\boldsymbol{\sigma}_0$ is the 2×2 unit matrix and det denotes the determinant. The characteristic equation can be written as

$$\nu_{\pm}^{4} - \mathrm{tr}\boldsymbol{\Omega}_{12}\nu_{\pm}^{2} + \det\boldsymbol{\Omega}_{12} = 0, \tag{9}$$

where tr stands for the trace. The (squared) singular values are then obtained as

$$\nu_{\pm}^{2} = \frac{1}{2}\mathrm{tr}\,\boldsymbol{\Omega}_{12}\left[1 \pm P_{\Omega}\left(\mathbf{r}_1,\mathbf{r}_2\right)\right], \tag{10}$$

where

$$P_\Omega(\mathbf{r}_1,\mathbf{r}_2) = \left(1 - \frac{4\det\mathbf{\Omega}_{12}}{\mathrm{tr}^2\mathbf{\Omega}_{12}}\right)^{1/2} \tag{11}$$

is bounded as $0 \le P_\Omega(\mathbf{r}_1,\mathbf{r}_2) \le 1$.

Next, we highlight some central properties of $\mathbf{\Omega}_{12}$ and $\mathbf{\Omega}_{21}$. Firstly, they are Hermitian and nonnegative definite matrices that satisfy the conditions $\mathbf{\Omega}_{12}^\dagger = \mathbf{\Omega}_{12}$, $\mathbf{\Omega}_{21}^\dagger = \mathbf{\Omega}_{21}$, $\mathrm{tr}\,\mathbf{\Omega}_{12} = \mathrm{tr}\,\mathbf{\Omega}_{21}$, $\det\mathbf{\Omega}_{12} = \det\mathbf{\Omega}_{21} \ge 0$, and contain nonnegative diagonal entries. Secondly, the coherence information in matrix $\mathbf{\Omega}_{12}$ is generally different from that in $\mathbf{\Omega}_{21}$ due to the quasi-Hermiticity of the CSD matrix as is evident from Eqs. (5) and (6). The mathematical properties of $\mathbf{\Omega}_{12}$ and $\mathbf{\Omega}_{21}$ are similar to those of the polarization matrix and they can formally be decomposed into two parts, one of which is proportional to the identity matrix and the other has zero determinant. This division is analogous to the decomposition of the polarization matrix into parts corresponding to a completely unpolarized beam and a fully polarized beam [1]. The earlier works concerning the coherence Poincaré sphere [4, 5] were extensively based on this division. Here we follow a different procedure and interpret the sphere using the SVD of the CSD only.

We proceed by defining the Stokes parameters of $\mathbf{\Omega}_{12}$ as

$$Q_j(\mathbf{r}_1,\mathbf{r}_2) = \mathrm{tr}(\boldsymbol{\sigma}_j\mathbf{\Omega}_{12}), \quad j = 0,\ldots,3, \tag{12}$$

where $\boldsymbol{\sigma}_j$, with $j \in (1,2,3)$, are the Pauli spin matrices [1]. We remark that analogous definitions hold naturally for $\mathbf{\Omega}_{21}$. These parameters are real-valued and contain information on the two-point spatial coherence of the beam. The Stokes parameters can be normalized as

$$q_j(\mathbf{r}_1,\mathbf{r}_2) = \frac{Q_j(\mathbf{r}_1,\mathbf{r}_2)}{S_0(\mathbf{r}_1)S_0(\mathbf{r}_2)}, \quad j = 0,\ldots,3, \tag{13}$$

with $S_0(\mathbf{r}) = \mathrm{tr}\,\mathbf{W}(\mathbf{r},\mathbf{r},\omega)$ being the spectral density of the beam. The parameters $q_j(\mathbf{r}_1,\mathbf{r}_2)$, $j = 1,2,3$, obey the quadratic equation

$$q_1^2(\mathbf{r}_1,\mathbf{r}_2) + q_2^2(\mathbf{r}_1,\mathbf{r}_2) + q_3^2(\mathbf{r}_1,\mathbf{r}_2) = P_\Omega^2(\mathbf{r}_1,\mathbf{r}_2)\mu^4(\mathbf{r}_1,\mathbf{r}_2), \tag{14}$$

where

$$\mu(\mathbf{r}_1,\mathbf{r}_2) = \left[\frac{\mathrm{tr}\,\mathbf{\Omega}_{12}}{S_0(\mathbf{r}_1)S_0(\mathbf{r}_2)}\right]^{1/2} \tag{15}$$

is the electromagnetic degree of coherence [3, 17]. This degree is bounded as $0 \le \mu(\mathbf{r}_1,\mathbf{r}_2) \le 1$, with the lower and upper bounds corresponding to complete incoherence and full coherence of the beam at points $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively.

Next we approach the geometric interpretation of the coherence Poincaré sphere in Eq. (14) via the singular value decomposition of the CSD. We first define the coherence Poincaré vector

$$\mathbf{q}(\mathbf{r}_1,\mathbf{r}_2) = \left[q_1(\mathbf{r}_1,\mathbf{r}_2), q_2(\mathbf{r}_1,\mathbf{r}_2), q_3(\mathbf{r}_1,\mathbf{r}_2)\right], \tag{16}$$

that displays the spatial coherence information in $\mathbf{\Omega}_{12}$ as points on or within a unit sphere in the (q_1,q_2,q_3) space. We remark that since the information content of $\mathbf{\Omega}_{12}$ is in general different from that of $\mathbf{\Omega}_{21}$, two coherence Poincaré vectors $\mathbf{q}_{12} = \mathbf{q}(\mathbf{r}_1,\mathbf{r}_2)$ and $\mathbf{q}_{21} = \mathbf{q}(\mathbf{r}_2,\mathbf{r}_1)$ are required to display the spatial coherence of the beam. Analytical expressions of these vectors are obtained from the SVD of the CSD matrix as is shown below. For this purpose we recall the unitarity conditions $\mathbf{U}^\dagger\mathbf{U} = \mathbf{V}^\dagger\mathbf{V} = \boldsymbol{\sigma}_0$, which together with the SVD and Eqs. (5) and (6) yield

$$\boldsymbol{\Omega}_{12} = \nu_+^2 \hat{\mathbf{v}}_+ \hat{\mathbf{v}}_+^\dagger + \nu_-^2 \hat{\mathbf{v}}_- \hat{\mathbf{v}}_-^\dagger, \tag{17}$$

$$\boldsymbol{\Omega}_{21} = \nu_+^2 \hat{\mathbf{u}}_+ \hat{\mathbf{u}}_+^\dagger + \nu_-^2 \hat{\mathbf{u}}_- \hat{\mathbf{u}}_-^\dagger. \tag{18}$$

Furthermore, unitarity of $\mathbf{U}$ and $\mathbf{V}$ implies that $\hat{\mathbf{v}}_+ \hat{\mathbf{v}}_+^\dagger + \hat{\mathbf{v}}_- \hat{\mathbf{v}}_-^\dagger = \boldsymbol{\sigma}_0$ and similarly for $\hat{\mathbf{u}}_\pm$. These together with Eqs. (10), (17), and (18) result in

$$\boldsymbol{\Omega}_{12} = \mathrm{tr}\,\boldsymbol{\Omega}_{12} \left(\frac{1 - P_\Omega}{2} \boldsymbol{\sigma}_0 + P_\Omega \hat{\mathbf{v}}_+ \hat{\mathbf{v}}_+^\dagger \right), \tag{19}$$

$$\boldsymbol{\Omega}_{21} = \mathrm{tr}\,\boldsymbol{\Omega}_{21} \left(\frac{1 - P_\Omega}{2} \boldsymbol{\sigma}_0 + P_\Omega \hat{\mathbf{u}}_+ \hat{\mathbf{u}}_+^\dagger \right), \tag{20}$$

where we have written $P_\Omega = P_\Omega(\mathbf{r}_1, \mathbf{r}_2)$. It is important to note that these two expressions coincide with the decompositions in Eq. (3) of [4] and Eq. (8) of [5] which constituted the starting point of the mentioned works without a reference to the SVD. Combining the equations above with the definitions in Eqs. (12), (13), (15), and (16), we find that

$$\mathbf{q}_{12} = P_\Omega \mu^2 \left[|v_{+x}|^2 - |v_{+y}|^2 , 2\mathrm{Re}(v_{+x} v_{+y}^*), 2\mathrm{Im}(v_{+x} v_{+y}^*) \right], \tag{21}$$

$$\mathbf{q}_{21} = P_\Omega \mu^2 \left[|u_{+x}|^2 - |u_{+y}|^2 , 2\mathrm{Re}(u_{+x} u_{+y}^*), 2\mathrm{Im}(u_{+x} u_{+y}^*) \right], \tag{22}$$

where $\mu = \mu(\mathbf{r}_1, \mathbf{r}_2)$. These expressions provide the singular-value interpretation of the two coherence Poincaré vectors that represent the state of spatial coherence of a partially coherent and partially polarized electromagnetic beam. Both vectors have the same length, $|\mathbf{q}_{12}| = |\mathbf{q}_{21}| = P_\Omega \mu^2$, and their directions are specified by the vectors $\hat{\mathbf{v}}_+$ and $\hat{\mathbf{u}}_+$. In addition, we note that the equalities $\mathrm{tr}\,\boldsymbol{\Omega}_{12} = \nu_+^2 + \nu_-^2$ and $\det \boldsymbol{\Omega}_{12} = \nu_+^2 \nu_-^2$ are obtained from Eq. (10), and by using them together with Eqs. (11) and (15) we see that $P_\Omega \mu^2 = (\nu_+^2 - \nu_-^2)/\left[S_0(\mathbf{r}_1) S_0(\mathbf{r}_2)\right]$. Hence, the length of the coherence Poincaré vectors can be viewed as the intensity-normalized distance between the squared singular values ν_+^2 and ν_-^2 of the CSD matrix, and their directions are specified by the singular vectors $\hat{\mathbf{v}}_+$ and $\hat{\mathbf{u}}_+$ corresponding to the larger singular value ν_+.

Next, we elucidate the physical meaning of the formalism. Firstly, for a completely coherent beam the degree of coherence equals unity, which yields $P_\Omega \mu^2 = 1$ [4] and hence the vectors $\mathbf{q}_{12}$ and $\mathbf{q}_{21}$ are unit-length vectors. Fully coherent beams are thus located on the surface of a unit sphere in the (q_1, q_2, q_3) space. Secondly, the origin is preserved for beams with $P_\Omega = 0$ or $\mu = 0$. The former includes the so-called pure unpolarized beams [18] and beams that can be transformed into such by a suitable unitary operation [5]. The latter naturally means that the beam is spatially fully incoherent.

For a fully polarized but spatially partially coherent beam $P_\Omega = 1$ holds [4], and the lengths of the coherence Poincaré vectors depend only on the degree of coherence as $|\mathbf{q}_{12}| = |\mathbf{q}_{21}| = \mu^2$. In addition, we note that the CSD matrix of a beam with an arbitrary state of full polarization can be written as $\mathbf{W}_{12} = W_{12} \hat{\mathbf{e}}_1^* \hat{\mathbf{e}}_2^{\mathrm{T}}$, where $W_{12} = \langle E^*(\mathbf{r}_1) E(\mathbf{r}_2) \rangle$ is a correlation function over an ensemble of random scalars $E(\mathbf{r}, \omega)$, and $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}(\mathbf{r}_n, \omega)$, $n = 1, 2$, are the deterministic Jones vectors that specify the polarization state of the beam at positions $\mathbf{r}_n$. As a consequence, the singular vectors are of the form $\hat{\mathbf{v}}_+ = \hat{\mathbf{e}}_2^*$ and $\hat{\mathbf{u}}_+ = \hat{\mathbf{e}}_1^*$. This implies that the coherence Poincaré vectors are expressible as $\mathbf{q}_{12} = \mu^2 \mathbf{s}_2$ and $\mathbf{q}_{21} =$

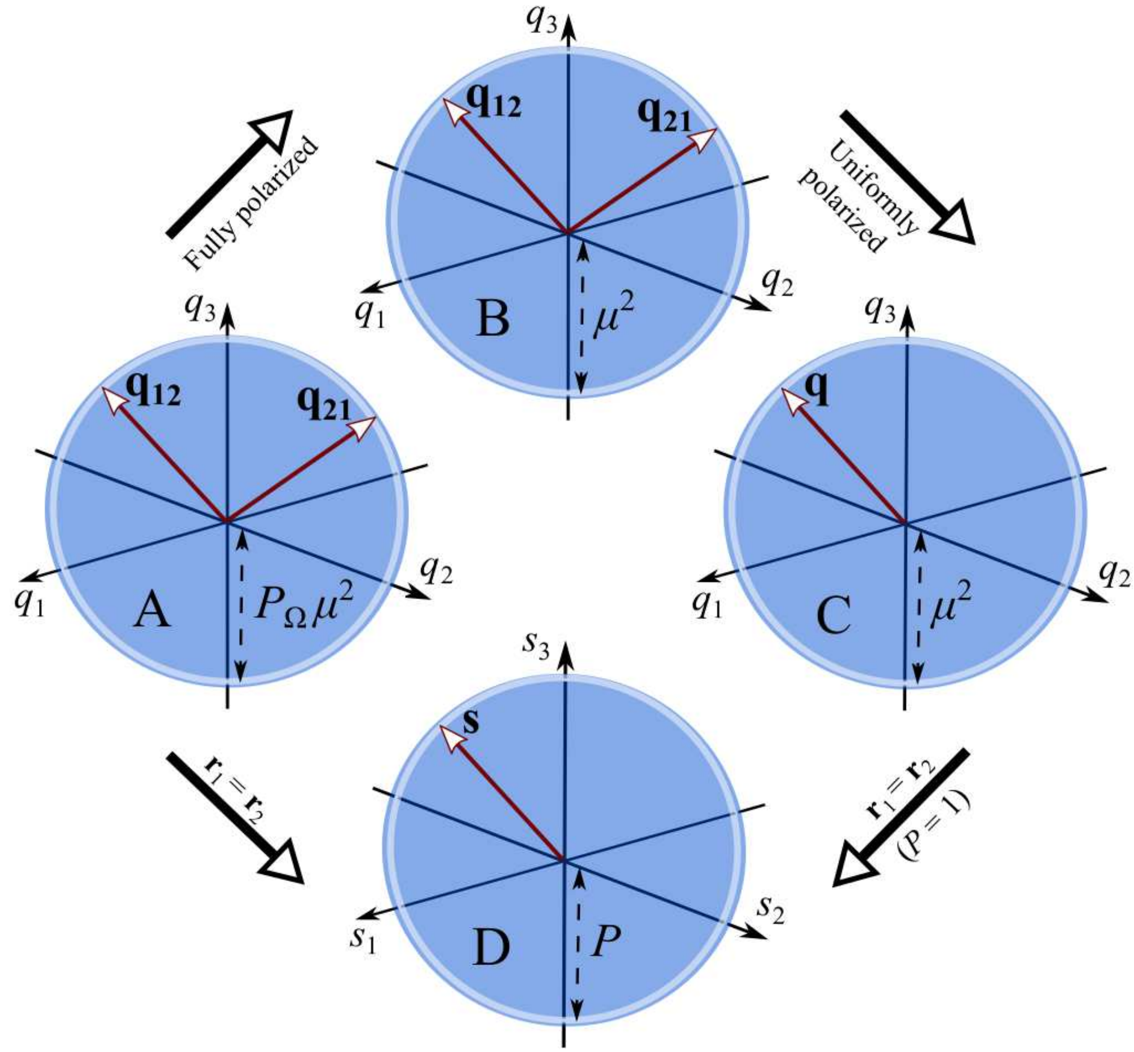

Figure 1

$\mu^2\mathbf{s}_1$, where $\mathbf{s}_n = [s_1(\mathbf{r}_n), s_2(\mathbf{r}_n), s_3(\mathbf{r}_n)]$ represent the polarization Poincaré vectors at $\mathbf{r}_n$ with $s_j(\mathbf{r}_n) = \mathrm{tr}\,(\boldsymbol{\sigma}_j\mathbf{W}_{nn})/S_0(\mathbf{r}_n)$, $j = 1, 2, 3$, being the (normalized) polarization Stokes parameters, $n = 1, 2$. In other words, for a fully polarized beam the directions of the coherence Poincaré vectors depict the state of polarization as in the context of the polarization Poincaré sphere. The coordinate axes in the (q_1, q_2, q_3) space represent the states of x, y, $\pm 45°$, right-hand, and left-hand circular polarization whereas elsewhere the beam is elliptically polarized. The vector $\mathbf{q}_{12}$ points out the polarization state of the beam at $\mathbf{r}_2$ and $\mathbf{q}_{21}$ does so at $\mathbf{r}_1$. Furthermore, if the state of polarization is uniform across the beam, these vectors converge into a single coherence Poincaré vector whose orientation expresses the uniform polarization state and length displays the squared degree of coherence at a pair of points. Finally, we observe that in a single point, $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$, the quantities $q_j(\mathbf{r}, \mathbf{r})$ reduce to the polarization Stokes parameters $s_j(\mathbf{r})$, $j = 1, 2, 3$, and the vector $\mathbf{q}(\mathbf{r}, \mathbf{r}) = \mathbf{s}(\mathbf{r})$ is the polarization Poincaré vector. Consequently, the traditional polarization Poincaré sphere [6, 7] is encountered when the formalism is applied at a single point. The various reductions described above are illustrated graphically in Fig. 1.

3 Conclusion

In summary, as an extension to the previous works [4, 5], we have derived the concept of the Poincaré sphere of electromagnetic two-point spatial coherence exclusively from the point of view of the singular value decomposition of the cross-spectral density matrix. The interpretation of the concept for an arbitrary partially polarized, partially spatially coherent beam follows directly from this approach; the state of coherence of the beam is depicted by two coherence Poincaré vectors whose lengths are defined by the normalized distance between the (squared) singular values of the CSD matrix and orientations are determined by the singular vectors related to the larger singular value. Furthermore, we highlighted the interpretation of this construction for fully polarized beams for which the coherence and polarization characteristics are closely linked, and noted that at a single point the formalism coincides with the traditional polarization Poincaré sphere.

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Competing interests

The authors declare that they have no competing interests.

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Figure 1: Poincaré sphere of electromagnetic spatial coherence. For an arbitrary partially polarized and partially coherent beam two coherence Poincaré vectors $\mathbf{q}_{12}$ and $\mathbf{q}_{21}$ with the same length $P_\Omega \mu^2$ but different orientations are generally needed (Sphere A). If the beam is fully polarized the lengths of these vectors display the degree of coherence, $|\mathbf{q}_{12}| = |\mathbf{q}_{21}| = \mu^2$, and their directions show the polarization state of the beam at points $\mathbf{r}_2$ and $\mathbf{r}_1$, respectively (Sphere B). If the beam is uniformly polarized, these vectors coincide and a single coherence Poincaré vector $\mathbf{q}$ is sufficient to represent the beam, with its length again showing the degree of coherence and the direction specifying the polarization state (Sphere C). At a single point the formalism reduces to the traditional polarization Poincaré sphere where the distance from the origin is given by the degree of polarization P [1] (Sphere D).