

# Quantum coherence and entanglement of the system of a five-level atom in the presence of nonlinear fields

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**Abstract.** We investigate the atom-field system in the framework of harmonic oscillators based on deformed Heisenberg algebras. We explore the dynamic characteristics of the considered system under the effect of a nonlinear field. In particular, we consider the atomic population, atomic coherence, and atom-field entanglement for a system that comprises a single five-level atom interacting with a single-mode nonlinear field when the deformation effect is taken into account. We examine the time evolution of the quantum quantifiers in the presence of deformation when the initial state of the quantized field is defined to be a nonlinear coherent state (CS) or a superposition state.

**Keywords:** Nonlinear fields, Quantum algebra, Deformed Heisenberg algebra, Superposition field state, Quantum correlation, Quantum coherence.

## 1 Introduction

In the realm of quantum optics, the model that sees the most widespread use is known as the Jaynes-Cummings model [1]. This model depicts how an atom reacts when it comes into contact with an electromagnetic field. A broad framework of a two-level atom interacting with a one-mode field with various kinds of nonlinearities has been developed [2]. In addition, many studies have been conducted on the interaction that occurs between a qubit and a two-mode cavity field [3–5]. The interaction between a three-level atom with various configurations ( $\Xi$ ,  $\Lambda$ , and  $V$ ) and a one- or two-mode field was studied [6–8]. Recent research has focused on investigating four-level systems in a variety of configurations to see how these systems interact with cavity modes [9]. In this context, a wide variety of four-level atomic system designs have been investigated and shown [10–17]. The subject of the interaction of five-level systems with M-type and K-type configurations has garnered a lot of attention in the domains of quantum optics and quantum information. The spontaneous emission in a five-level atomic system driven by four fields [18], the phase-dependent optical properties of five-level K-type

atom systems [19], the effect of decoherence in a five-level atom system with  $\Xi$ -configuration [20], and the controllable entanglement of light in a five-level system [21] are some of the optical phenomena that have been investigated. On the other hand, a preliminary analysis of populations in a five-level atom with various photon distributions was considered [22].

Many quantum phenomena have been produced as resources for carrying out numerous tasks in the field of quantum optics and information, both theoretically and practically. This development took place over the course of a few years. In most cases, the coherent superposition of quantum states is regarded as an essential need for the existence of quantum correlations [23–29]. The idea of the EPR paradox [30] that was presented by Einstein, Podolsky, and Rosen was the impetus behind the establishment of this precondition. They described how the theory of quantum physics enables spooky activity at a distance. Schrödinger viewed the problem as one in which it was possible for local observations to influence a distant system even if he did not have access to it [31]. After that, Bell came up with the so-called Bell inequality as a way to demonstrate that this “spooky” phenomenon was not a coincidence. Subsequently, Bell presented the so-called Bell inequality to demonstrate that this “spooky activity” causes

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a quantum correlation that defies any classical explanation [32]. This was done in order to prove that the existence of this quantum correlation cannot be explained using conventional methods. Quantum coherence is the fundamental concept that underpins a variety of quantum phenomena that occur in nanomaterials [33, 34], quantum measurements and quantum metrology [35–39], and applications of quantum mechanics to biological things [40–42]. In line with the fundamental significance of quantum coherence, it was not until recently that an accurate theory of coherence was emphasized with the constraints necessary to ensure that quantum coherence is a physical resource [43]. This was done in light of the fact that quantum coherence has been given increasing attention in recent years. As a result, many quantum measures that check these limitations have been developed, most notably those that are based on the  $l_1$  norm and relative entropy [43]. In addition, the quantum coherence may be measured by either the convex-roof construction or nonlocal correlation [44, 45], and an operational theory of coherence has been developed [46].

Quantum entanglement is the aspect of a composite quantum system that most captivates my attention as a fascinating occurrence. If the combined state of two particles cannot be represented as a product of the states of their constituent subsystems, then the particles are said to be entangled [30]. A deeper comprehension of fundamental quantum processes may be attained through the production and manipulation of these entangled states. Examples of complicated entangled states that are utilized in tests of quantum nonlocality include triplets of particles named after Greenberger, Horne, and Zeilinger [47, 48]. In addition to these fundamental qualities, entanglement has developed into a crucial resource in quantum information processing, which is a field that has made significant strides in the past few years [49, 50]. In the present manuscript, we investigate the atom-field system in the framework of harmonic oscillators based on deformed Heisenberg algebras. We explore the dynamic characteristics of the considered system under the effect of a nonlinear field. We consider a system that comprises a single five-level atom interacting with a single-mode nonlinear field when the deformation effect is considered. We analyze the time evolution of the atomic population, atomic coherence, and atom-field entanglement in the presence of deformation when the initial state of the quantized field is defined to be a nonlinear CS or a superposition state.

The remaining parts of the article are structured as follows. In Section 2, a model of a five-level cascade-type atom and one-mode system is shown to the reader. In addition to that, the wave function of the atomic system under consideration is shown below. The quantum quantifiers related to the atom-field system and the results of numerical simulations are discussed in Section 3. In Section 4, a summary and some conclusions are presented.

## 2 Physical model and dynamics

The states of the oscillator are characterized using Schrödinger cat states in deformed spaces. We choose an deformed algebra to define the Hamiltonian of an

f-deformed oscillator as follows, assuming that the oscillator has a unitary frequency,

$$H = \frac{\hbar}{2} (\widehat{C}^+ \widehat{C} + \widehat{C} \widehat{C}^+), \quad (1)$$

where the operators  $\widehat{C}$  and  $\widehat{C}^+$  are considered to obey the following commutator algebras

$$[\widehat{C}, \widehat{n}] = \widehat{C}, \quad [\widehat{C}^+, \widehat{n}] = -\widehat{C}^+. \quad (2)$$

As for the standard operators of bosons that are non-deformed, it is not necessary for  $\widehat{C}$  and  $\widehat{C}^+$  to be connected to photon number operator  $\widehat{n}$  in the conventional manner; that is, in general,  $\widehat{C}^+ \widehat{C} = [\widehat{n}] \neq \widehat{n}$ , where is called a box function that determines the field nonlinearity that  $CC^+ = [n + 1]$ . The definition of the commutation relation of operators  $\widehat{C}$  and  $\widehat{C}^+$  is given as follows:

$$\widehat{C} \widehat{C}^+ - \widehat{C}^+ \widehat{C} = [\widehat{n} + 1] - [\widehat{n}]. \quad (3)$$

It is possible to see the deformed oscillators as particular examples of the so-called  $g$ -oscillators, in which the  $g$ -oscillator operators are specified as

$$\widehat{C} = \widehat{c}g(\widehat{n}) = g(\widehat{n} + 1)\widehat{c}, \quad (4)$$

$$\widehat{C}^+ = g(\widehat{n})\widehat{c}^\dagger = \widehat{c}^\dagger g(\widehat{n} + 1), \quad (5)$$

and that

$$[\widehat{C}, \widehat{C}^+] = (\widehat{n} + 1)g^2(\widehat{n} + 1) - \widehat{n}g(\widehat{n}), \quad (6)$$

where  $g(\widehat{n})$  represents an operator function of the Hermitian number operator  $\widehat{n} = \widehat{c}^+ \widehat{c}$ . The nonlinearity results from the dependence variable  $g$ . In the  $g \rightarrow 1$  limit, the deformed algebra will be reduced to the Heisenberg algebra.

$$[\widehat{c}, \widehat{c}^+] = 1, \quad [n, \widehat{c}] = -\widehat{c} \quad \text{and} \quad [n, \widehat{c}^+] = \widehat{c}^+. \quad (7)$$

The operators  $\widehat{C}$  and  $\widehat{C}^+$  act on the Fock states as:

$$\widehat{C}^+ |n\rangle = \sqrt{[n+1]_g} |n+1\rangle, \quad \widehat{C} |n\rangle = \sqrt{[n]_g} |n-1\rangle. \quad (8)$$

where  $|n\rangle$  form an orthonormal basis in the deformed Hilbert space.

In comparison to the Glauber states, the  $g$ -CSs are defined in terms of eigenvectors of the deformed operator  $\widehat{C}$

$$\widehat{C} |\xi, g\rangle = \xi |\xi, g\rangle, \quad (9)$$

where  $\xi$  is a complex eigenvalue of  $\widehat{C}$ . The deformed CSs are formulated as

$$|\xi, g\rangle = \mathcal{N}_g \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{[n]_g!}} |n\rangle, \quad \mathcal{N}_g = [\exp_g[|\xi|^2]]^{-\frac{1}{2}}, \quad (10)$$

where we have introduced

$$\exp_g[x] = \sum_{n=0}^{\infty} \frac{x^n}{[n]_g!}, \quad [n]_g! = [ng^2(n)] \times [(n-1)g^2(n-1)] \times \dots \times [g^2(1)]. \quad (11)$$

The function  $\exp_g$  is a deformed version of the usual exponential function with  $\exp_g[x]\exp_g[y] \neq \exp_g[x + y]$  and they coincide when the function  $g$  tends to the value one. Here, we use the effect of  $q$ -deformation of the box function [51]

$$g(n) = \left[ \frac{1}{n} \frac{1 - q^{-n}}{q - 1} \right]^{\frac{1}{2}}, \quad q \in \mathbb{R}. \quad (12)$$

It is obvious that  $g(n) = 1$  when  $q = 1$ , thus in this circumstance, nonlinear CSs become the standard CSs.

Now, we consider a five-level atom in the form of ladder-type, as illustrated in Figure 1, with energy  $\omega_j$  ( $j = 1, 2, 3, 4, 5$  and  $\omega_1 > \omega_2 > \dots > \omega_5$ ) as the energies of transition through each of the five level of the atom. The frequency and the creation (annihilation) operator  $\hat{a}^+$  ( $\hat{a}$ ) are the two variables that are used to characterize the field mode. The total Hamiltonian  $\hat{H}$  for the system that is being investigated is expressed in the rotating wave approximation as

$$\hat{H} = \hat{H}_{A-F} + \hat{H}_{IN}, \quad (13)$$

where the Hamiltonian for the free field and the non-interacting atom is denoted by the  $\hat{H}_{A-F}$ , and the interaction component of the total Hamiltonian is given by

$$\hat{H}_{A-F} = \sum_j \omega_j \hat{\sigma}_{j,j} + \Omega \hat{C}^\dagger \hat{C}. \quad (14)$$

where  $\hat{\sigma}_{j,j} = |j\rangle\langle j|$ .

In the meanwhile, we take into consideration the possibility that the connection between the atomic system and the field is mediated by four-photon processes. These are transitions that need the use of four photons to complete. The interaction Hamiltonian of the system is formulated as

$$\hat{H}_{IN} = \sum_{k=1}^4 \beta_k \left[ \hat{C} \hat{\sigma}_{k,k+1} + h.c. \right], \quad (15)$$

where  $\beta_k$  is the atom-field coupling constant, and we consider the case of identical coupling (i.e.,  $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta$ ).

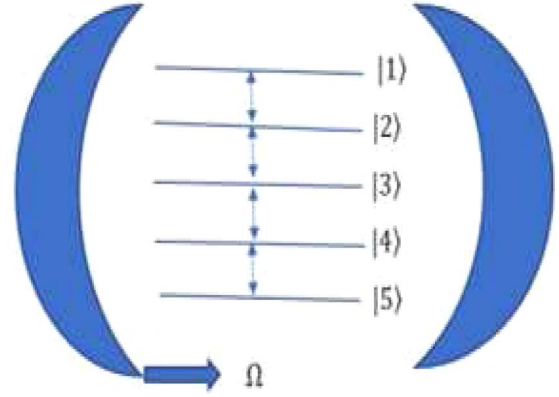
We assume that the five-level atom starts in its upper state  $|1\rangle$  and the field from the superposition of the  $g$ -CS. The initial state of the whole system is

$$\begin{aligned} |U(0)\rangle &= \frac{1}{\sqrt{1 + r^2 + 2r\mathcal{N}_g^2 \exp_g[-|\xi|^2]}} (|\xi, g, 1\rangle + r| - \xi, g, 1\rangle) \\ &= \sum_n \varpi_n |n, 1\rangle. \end{aligned} \quad (16)$$

The atom-field wave function at an arbitrarily scaled time  $T = \beta t$  is written in the form

$$|U(T)\rangle = \sum_n \varpi_n \left[ \sum_{k=1}^5 u_j(T) |n + k - 1\rangle \otimes |k\rangle \right], \quad (17)$$

where  $\varpi_n = \exp\left(\frac{-\bar{n}}{2}\right) \frac{\alpha^n}{\sqrt{n!}}$  in the limit  $q \rightarrow 1$  and  $r = 0$  with  $\bar{n} = |\alpha|^2$  is the initial mean photon number for the field mode.



**Fig. 1.** Schematic diagram of a five-level ladder-type atom interacting with a single-mode nonlinear field.

The time-dependent coefficients  $u_i$  are determined through solving the system of differential equations obtained from the Schrödinger equation

$$i \frac{d}{dT} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 & B_1 & 0 & 0 & 0 \\ B_1 & 0 & B_2 & 0 & 0 \\ 0 & B_2 & 0 & B_3 & 0 \\ 0 & 0 & B_3 & 0 & B_4 \\ 0 & 0 & 0 & B_4 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}, \quad (18)$$

where

$$B_k = \beta \sqrt{n + k} \quad k = 1, 2, 3, 4.$$

### 3 Quantum quantifiers and numerical results

This section reviews the basic definitions and concepts of quantumness measures considering the quantum coherence and entanglement. Moreover, we present and discuss the obtained results.

The fundamental properties of coherence depend on the diagonal elements of the system density operator. The  $l_1$  norm of quantum coherence is a measure of coherence through consideration of the absolute value of the non-diagonal elements. The measure of coherence based on entropy is defined as the distance between the state of interest and the closest incoherent state. The  $l_1$  norm of coherence is introduced as

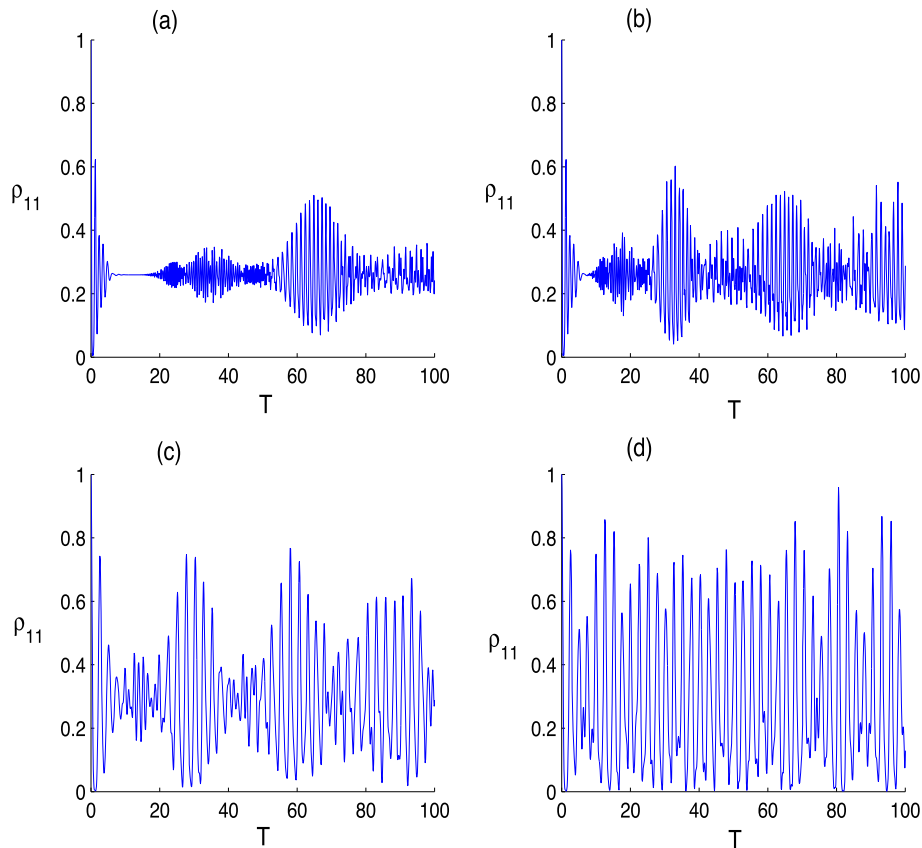
$$C_l = \min_{\delta \in \mathcal{I}} \|\rho - \delta\|_{l_1} = \sum_{i \neq j} |\rho_{ij}|, \quad (19)$$

where  $\mathcal{I}$  is the set of incoherent states and  $i$  and  $j$  are, respectively, the row and column indices. In terms of the von Neumann entropy function

$$S(\rho) = -\text{tr}(\rho \ln \rho), \quad (20)$$

the relative entropy coherence can be provided as

$$C_R = S(\rho || \rho_{\text{diag}}) = S(\rho_{\text{diag}}) - S(\rho), \quad (21)$$



**Fig. 2.** Time evolution of the upper state populations  $\rho_{11}$  of a five-level system for the field initially in (a) a CS with  $\alpha = 5$  and  $q = 1$ , (b) a superposition of CS with  $\alpha = 5$  and  $q = 1$ , (c) a CS with  $\alpha = 5$  and  $q = 1.5$  and (d) a superposition of CS with  $\alpha = 5$  and  $q = 1.5$ .

where the operator  $\rho_{\text{diag}}$  describes the quantum incoherent state.  $C_{l_1}$  and  $C_r$  both achieve monotony for all quantum states. It has been shown that  $C_{l_1}$  characterizes the upper bound of  $C_r$  for a pure state.

To quantify the entanglement, we use the von Neumann function. This function value gets zero value for all pure states, which are states that fulfill the condition  $\hat{\rho}^2 = \hat{\rho}$  with  $\hat{\rho}$  is the density operator that describes a particular quantum system. In our situation, the entanglement of the atom (5LA)-field (F) state can be determined through the von Neumann entropy of one of subsystems

$$S_{5\text{LA}[\text{F}]} = -\text{Tr}_{\text{A}[\text{F}]} \{ \hat{\rho}_{5\text{LA}[\text{F}]} \ln [ \hat{\rho}_{5\text{LA}[\text{F}]} ] \} = - \sum_j \zeta_j \ln \zeta_j, \quad (22)$$

where the reduced density operators  $\hat{\rho}_{\text{A}[\text{F}]}$  are given by

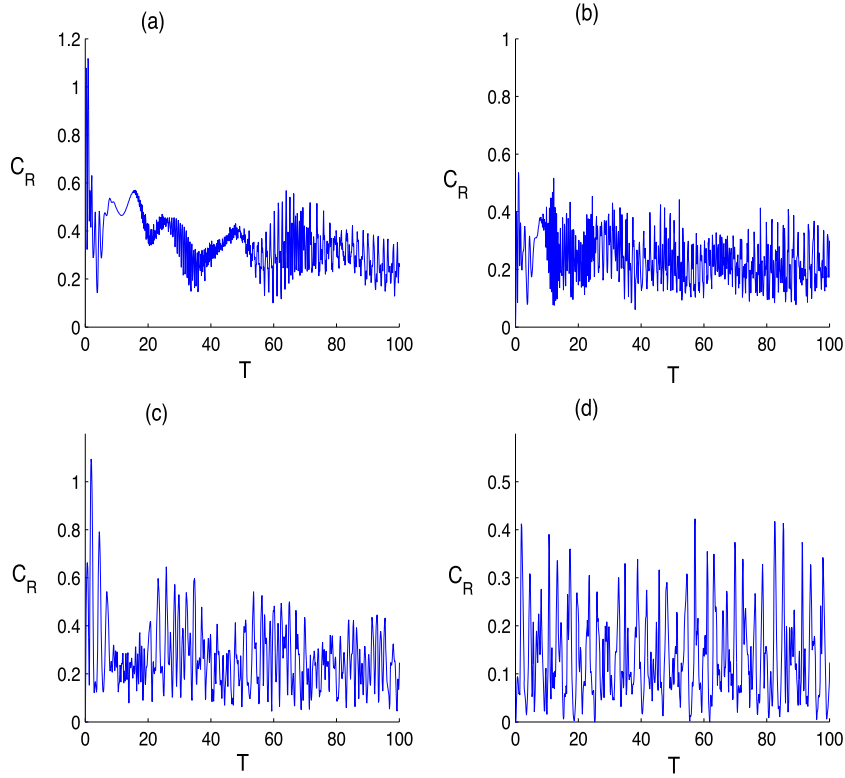
$$\hat{\rho}_{5\text{LA}[\text{F}]} = \text{Tr}_{\text{A}[\text{F}]} \{ |U(T)\rangle \langle U(T)| \}. \quad (23)$$

and  $\zeta_j$  denotes the eigenvalues of the atomic reduced density matrix (22).

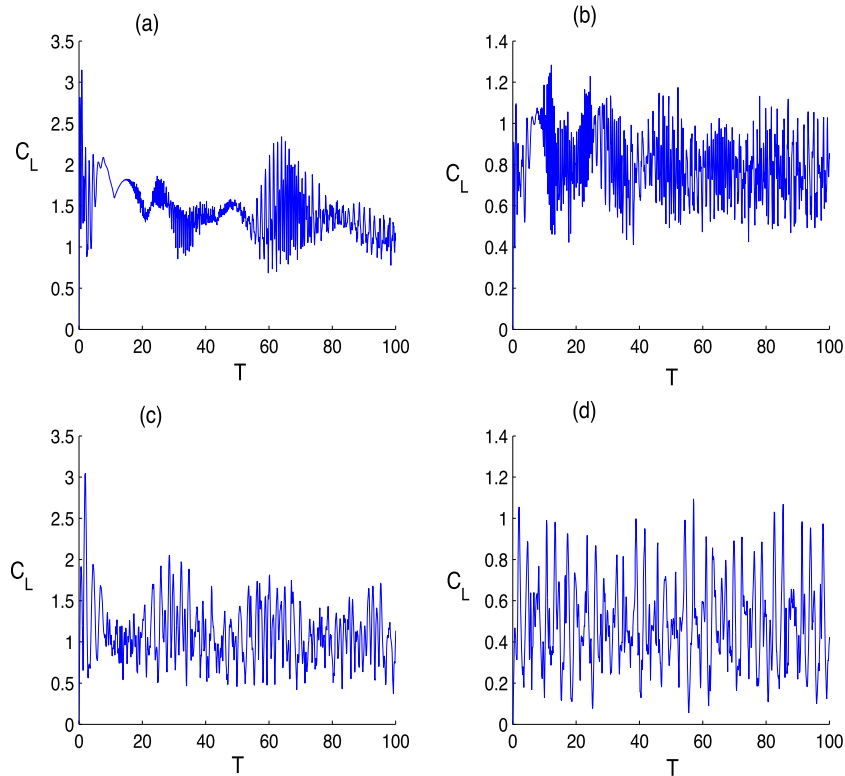
We now examine the effects of the initial state of the field and deformation parameter on the population, von Neumann entropy, and quantum coherence in the dynamics of the considered system. The numerical results are presented in Figures 1–4. We assume that the initial field is in a CS or a superposition state with  $\alpha = 5$ .

Figure 2 presents the variation in the population against the dimensionless time  $\beta t$  when the five-level atom starts from the upper state and the input field is defined in a CS or a superposition state for different values of the parameter  $q$ . Figures 2a and 2b, respectively, presents the results for  $r = 0$  and  $r = 1$  with  $q = 1$ . Figures 2c and 2d, respectively, presents the results for  $r = 0$  and  $r = 1$  with  $q = 1.5$ . Generally, the dynamics of the function  $\rho_{11}$  exhibits fast oscillations with amplitudes depending on  $q$  and  $r$ . These parameters may have a significant impact on the control of the inhibition decay of the excited state. Figure 2 shows that the presence of deformation increases the amplitude of the function  $\rho_{11}$  and reduces its oscillation. Furthermore, the single-mode field with  $r = 1$  increases the amplitude of the population.

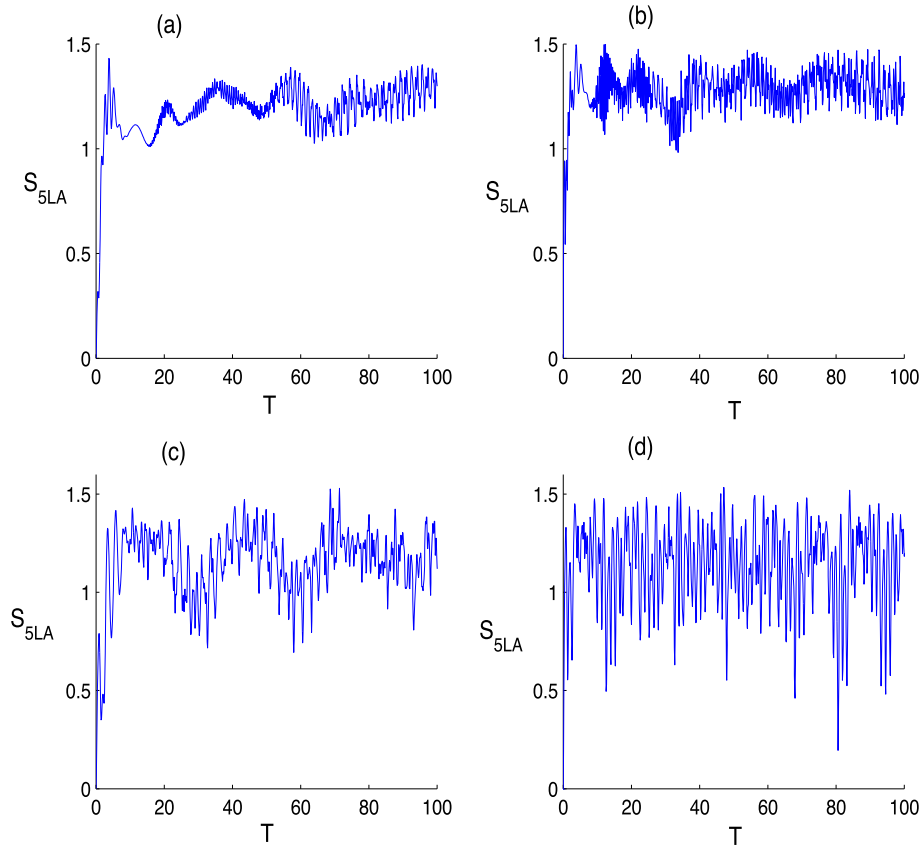
When considering a five-level atom that was first initialized in its upper state, the dynamics of the quantum coherence in the atomic system are analyzed when the single-mode field is specified by a CS or a superposition of CS that depend on  $q$  and  $r$ . Figures 3 and 4 show the dynamics of the functions  $C_R$  and  $C_L$  versus the dimensionless time  $T$  for different values of  $q$  with  $r = 0$  and  $r = 1$ . In general, the quantum coherence of the five-level atom largely depends on the initial form of the field and its deformation. Figures 3 and 4 show that the quantum coherence of the atom can be generated through the field-atom



**Fig. 3.** Time evolution of the quantum coherence  $C_R$  of a five-level system for the field initially in (a) a CS with  $\alpha = 5$  and  $q = 1$ , (b) a superposition of CS with  $\alpha = 5$  and  $q = 1$ , (c) a CS with  $\alpha = 5$  and  $q = 1.5$  and (d) a superposition of CS with  $\alpha = 5$  and  $q = 1.5$ .



**Fig. 4.** Time evolution of the quantum coherence  $C_L$  of a five-level system for the field initially in (a) a CS with  $\alpha = 5$  and  $q = 1$ , (b) a superposition of CS with  $\alpha = 5$  and  $q = 1$ , (c) a CS with  $\alpha = 5$  and  $q = 1.5$  and (d) a superposition of CS with  $\alpha = 5$  and  $q = 1.5$ .



**Fig. 5.** Time evolution of the von Neumann entropy  $S_{5LA}$  of a five-level system for the field initially in (a) a CS with  $\alpha = 5$  and  $q = 1$ , (b) a superposition of CS with  $\alpha = 5$  and  $q = 1$ , (c) a CS with  $\alpha = 5$  and  $q = 1.5$  and (d) a superposition of CS with  $\alpha = 5$  and  $q = 1.5$ .

interaction in the time dynamics. It seems that the functions  $C_R$  and  $C_L$  have oscillatory behavior with the amplitude depending on the parameters  $r$  and  $q$ . Such input fields can thus help control the quantum coherence of the atomic system in the time dynamics, where the functions  $C_R$  and  $C_L$  have high amplitudes depending on the parameter  $q$  for  $r = 0$  and  $r = 1$ . Moreover, Figures 3 and 4 show that the presence of deformation reduces the oscillations of the functions  $C_R$  and  $C_L$  and stabilizes their behavior with the time. Meanwhile, the superposition of the CSs for  $r = 1$  enhances the oscillation of the functions  $C_R$  and  $C_L$  with more random behavior in the dynamics. Finally, Figure 5 presents the dynamics of the von Neumann entropy for the same conditions on the physical parameters of the quantum system. It is clear that the enhancement of entanglement of the atom-field state in the dynamics can be improved by a suitable choice of the deformed parameter for both cases  $r = 0$  and  $r = 1$ . Furthermore, the entropy function behaves in a similar way as the quantum coherence with respect to the parameters of the quantum physics. Coherence is thus a good candidate for describing the non-local correlation, and its implementation in different areas of quantum information and optics is promising. The physical origin of the different behaviors of the quantum coherence might be the flow of information between the atom and nonlinear field as the correlations established during the interaction. In short, the results demonstrate that the

quantum coherence and nonlocal correlation can be controlled and maintained through the deformation of the coherent field during the atom-field interaction and its initial state setting for describing factual experimental scenarios under optimal conditions.

## 4 Conclusion

We have investigated the atom-field system within the framework of harmonic oscillators based on deformed Heisenberg algebras. We have explored the dynamic characteristics of the considered system under the effect of a nonlinear field. In particular, we have considered the atomic population, atomic coherence, and atom-field entanglement for a system that comprises a single five-level atom interacting with a single-mode nonlinear field when the deformation effect is taken into account. We have examined the time evolution of the quantum quantifiers in the case of deformation when the initial state of the quantized field is defined to be a nonlinear CS or superposition of CS. The results demonstrated that the quantum coherence and nonlocal correlation can be controlled and maintained through the deformation of the coherent field during the atom-field interaction and its initial state setting for describing factual experimental scenarios under optimal conditions. Here, we



have considered the case of a closed systems, then it is very useful to introduce in the future the environment effect on the coherence and correlations considering for the case of open quantum systems with large levels.

## Conflict of interest

The authors declare that there is no conflict of interest.

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