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SHORT COMMUNICATION · OPEN ACCESS

It is a sufficient condition only, not a necessary and sufficient condition, for decomposing wavefront aberrations

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Abstract. The classic equation for decomposing the wavefront aberrations of axis-symmetrical optical systems has the form,

$$W(h_0, \rho, \phi) = \sum_{j=0}^{\propto} \sum_{p=0}^{\propto} \sum_{m=0}^{\propto} C_{(2j+m)(2p+m)m} (h_0)^{2j+m} (\rho)^{2p+m} (\cos\phi)^m$$

where j, p and m are non-negative integers, ρ and ϕ are the polar coordinates of the pupil, and h_0 is the object height. However, one non-zero component of the aberrations (i.e., $C_{133} h_0 \rho^3 \cos^3\phi$) is missing from this equation when the image plane is not the Gaussian image plane. This implies that the equation is a sufficient condition only, rather than a necessary and sufficient condition, since it cannot guarantee that all of the components of the aberrations can be found. Accordingly, this paper presents a new method for determining all the components of aberrations of any order. The results show that three and six components of the secondary and tertiary aberrations, respectively, are missing in the existing literature.

Keywords: Aberrations, Imaging systems, Wavefront.

1 Introduction

The wavefront and ray aberrations of axis-symmetrical systems have attracted significant attention in the literature [1–12]. The usual equation for decomposing the monochromatic wavefront aberration $W(h_0, \rho, \phi)$ into different orders and components is given as (e.g., Eq. (3.31b) of [5]),

$$W(h_0, \rho, \phi) = \sum_{j=0}^{\propto} \sum_{p=0}^{\propto} \sum_{m=0}^{\propto} C_{(2j+m)(2p+m)m} (h_0)^{2j+m} (\rho)^{2p+m} (\cos\phi)^m \tag{1}$$

where j, p and m are non-negative integers, ρ and ϕ are the polar coordinates of the pupil, and h_0 is the object height. The sum of the powers of h_0 and ρ gives the order of the related component. That is,

$$2q = 2(j + p + m) \tag{2}$$

For example, if the piston term is included, the primary (i.e., fourth-order $W_{4\text{th}}$) aberrations are obtained from equation (1) with $2q = 4$ as,

$$W_{4\text{th}} = C_{400} h_0^4 + C_{040} \rho^4 + C_{131} h_0 \rho^3 \cos\phi + C_{222} h_0^2 \rho^2 (\cos\phi)^2 + C_{220} h_0^2 \rho^2 + C_{311} h_0^3 \rho \cos\phi \tag{3}$$

where the six components of the equation represent the piston term and the spherical, coma, astigmatism, field curvature, and distortion aberrations, respectively. The composition ability of the primary aberrations from equation (1) is echoed by Buchdahl [1], who computed the Buchdahl aberration coefficients to determine the wavefront and ray aberrations of axis-symmetrical systems. However, the question arises as to whether equation (1) provides all the components of the various order wavefront aberrations in an axis-symmetrical optical system.

2 Decomposition of wavefront aberrations

Equation (1) is based on the fact that the aberration function $W(h_0, \rho, \phi)$ must satisfy the following three equations related to the fundamental axis-symmetrical nature of axis-symmetrical systems (e.g., p. 154 of [5]):

$$W(0, \rho, \phi) = W(0, -\rho, \phi) \tag{4}$$

$$W(h_0, \rho, \phi) = W(h_0, \rho, -\phi) \tag{5}$$

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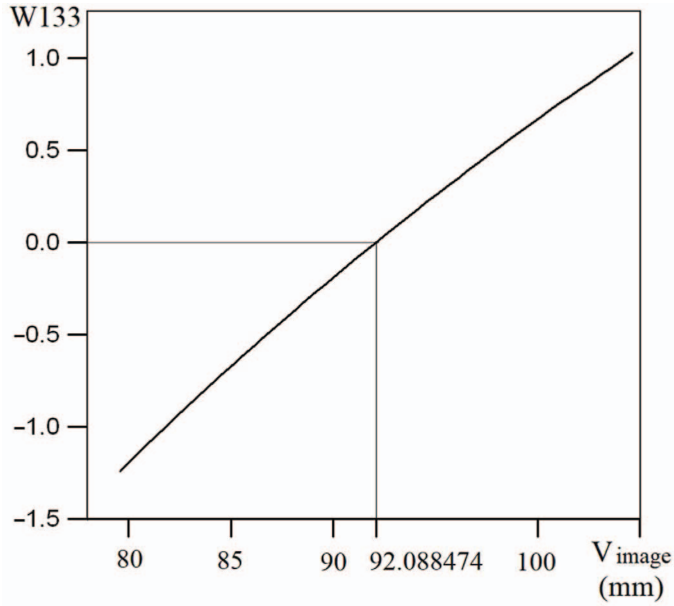

Fig. 1. The variation of $W_{133} = C_{133}h_0\rho^3/\lambda$ (where $h_0 = 17$ mm, $\rho = 21$ mm, and $\lambda = 550$ μm) versus the separation of image plane V_{image} for the optical system of [12]. The Gaussian image plane of this system is located at $V_{\text{image}} = 92.088474$ mm when the object is placed at $P_{0z} = -200$ mm. This figure shows that $C_{133}h_0\rho^3(\cos\phi)^3$ has non-zero value when the image plane is not the Gaussian image plane. It also shows that $C_{133} = 0$ when $V_{\text{image}} = 92.088474$ mm, indicating $C_{133}h_0\rho^3(\cos\phi)^3$ is the defocus component of primary aberrations.

$$W(h_0, \rho, \phi) = W(-h_0, \rho, \pi + \phi) = W(-h_0, \rho, \pi - \phi) \quad (6)$$

Equation (4) indicates that the aberration function of an on-axis object must be radially symmetric, and hence implies that the components of $W(h_0, \rho, \phi)$ that do not depend on h_0 should vary as ρ^2 (or its integer power). Equation (5) states that $W(h_0, \rho, \phi)$ must be a function of $\cos\phi$. Finally, equation (6) shows that $W(h_0, \rho, \phi)$ must equal $W(-h_0, \rho, \pi + \phi)$ for an object with height h_0 above the optical axis and $W(-h_0, \rho, \pi - \phi)$ for an object with height h_0 below the optical axis. Hence, those terms that depend on ϕ should be a function of $h_0\rho\cos\phi$. Combining this with ϕ-independent terms, it follows that $W(h_0, \rho, \phi)$ must consist of terms containing h_0^2, ρ^2 and $h_0\rho\cos\phi$ factors, to have a sufficient condition given by equation (1). Note that a sufficient condition is taken here to mean that any term generated by equation (1) is a component of an aberration.

The present group recently proposed a method for determining the aberrations of axis-symmetrical optical systems [12]. It is shown in Figure 1 that when the image plane is not the Gaussian image plane, a non-zero component (i.e., $C_{133}h_0\rho^3(\cos\phi)^3$) is missing from equation (1), even though it satisfies equations (4)–(6). This implies that equation (1) alone does not guarantee that all of the components of the aberrations can be found. In other words, equation (1) is not a necessary and sufficient condition for determining all the components of the aberrations in an axis-symmetrical optical system.

Thus, a second question arises as to how all of these components may be found. To address this question, it is first necessary to realize that the power of $\cos\phi$ should be non-negative. That is,

$$m \geq 0 \quad (7)$$

Without any loss of generality, the power of h_0 can be confined to a non-negative integer value in order to have $2j + m \geq 0$. That is,

$$j \geq -m/2 \quad (8)$$

Mathematically, the power of ρ should be greater than or equal to the power of $\cos\phi$, i.e., $2p + m \geq m$, which yields,

$$p \geq 0 \quad (9)$$

One then has the following inequality from the sum of $q = j + p + m$ (see Eq. (2)) and equation (8):

$$p \leq q - m/2 \quad (10)$$

The intersection of equations (9) and (10) defines the possible range of p. That is,

$$0 \leq p \leq q - m/2 \quad (11)$$

Equation (11) shows that the integer index p starts at $p = 0$ and ends at,

$$p_{\max} = \langle q - m/2 \rangle \quad (12)$$

where $\langle q - m/2 \rangle$ is the maximum non-negative integer value of p for a given m and q. In other words, index p belongs to the following set:

$$p \in \{0, 1, 2, ..., p_{\max}\} \quad (13)$$

Table 1. Components of secondary aberrations with $q = 3$ in axis-symmetrical system.

m	p	$(2q - 2p - m, 2p + m, m)$	Aberration component
		$q = 3$	
0	0	(6, 0, 0)	$C_{600}h_0^6$
	1	(4, 2, 0)	$C_{420}h_0^4\rho^2$
	2	(2, 4, 0)	$C_{240}h_0^2\rho^4$
	3	(0, 6, 0)	$C_{060}\rho^6$
1	0	(5, 1, 1)	$C_{511}h_0^5\rho\cos\phi$
	1	(3, 3, 1)	$C_{331}h_0^3\rho^3\cos\phi$
	2	(1, 5, 1)	$C_{151}h_0\rho^5\cos\phi$
2	0	(4, 2, 2)	$C_{422}h_0^4\rho^2(\cos\phi)^2$
	1	(2, 4, 2)	$C_{242}h_0^2\rho^4(\cos\phi)^2$
	2	(0, 6, 2)	No
3	0	(3, 3, 3)	$C_{333}h_0^3\rho^3(\cos\phi)^3$
	1	(1, 5, 3)	$C_{153}h_0\rho^5(\cos\phi)^3$
4	0	(2, 4, 4)	$C_{244}h_0^2\rho^4(\cos\phi)^4$
	1	(0, 6, 4)	No
5	0	(1, 5, 5)	$C_{155}h_0\rho^5(\cos\phi)^5$
6	0	(0, 6, 6)	No

Furthermore, from equation (11), the possible upper limit of m which yields $0 \leq q - m/2$ is,

$$m \leq 2q \tag{14}$$

The intersection of equations (7) and (14) then shows the possible domain of integer m for a given q. That is,

$$0 \leq m \leq 2q \tag{15}$$

Given the preceding derivations, it is possible to obtain all components of any order (say, the $(2q)$th order) wavefront aberration for an axis-symmetrical optical system by the following equation when $j = q - (p + m)$ is used:

$$W_{(2q)\text{th-order}}(h_0, \rho, \phi) = \sum_{p=0}^{p=p_{\max}} \sum_{m=0}^{m=2q} C_{(2q-2p-m)(2p+m)m}(h_0)^{2q-2p-m}(\rho)^{2p+m}(\cos\phi)^m \tag{16}$$

Note that, as shown in equation (4), components of the aberrations that do not depend on h_0 should vary as ρ^2, or as its integer power only. In other words, if $2q - 2p - m = 0$ and $m \neq 0$, that component generated from equation (16) does not exist.

Consider Table 1 below, which shows all the components of the secondary aberration ($q = 3$) of an axis-symmetrical optical system for illustration purposes. The entries of the first and second columns are the values of m and p obtained from equations (15) to (13), respectively. Meanwhile, the entries of the third column denote the sequence $(2j + m, 2p + m, m) = (2q - 2p - m, 2p + m, m)$ for each value of m. The fourth column shows the aberration component for each value of m (if it exists). Comparing Table 1 with the existing literature, it is found that three components in Table 1 (i.e., $C_{244}h_0^2\rho^4(\cos\phi)^4$, $C_{153}h_0\rho^5(\cos\phi)^3$ and $C_{155}h_0\rho^5(\cos\phi)^5$) are not included among the secondary aberrations given in the literature despite satisfying equations (4)–(6). In order to validate Table 1, the methodology proposed in [12] was extended to determine the values of all the secondary aberrations listed in the right-hand column of the table [3]. The results confirmed that all of the secondary aberrations possessed non-zero values.

Table 2. Components of tertiary aberrations with $q = 4$ in axis-symmetrical system.

m	p	$(2q - 2p - m, 2p + m, m)$	Aberration component
		$q = 4$	
0	0	(8, 0, 0)	$C_{800}h_0^8$
	1	(6, 2, 0)	$C_{620}h_0^6\rho^2$
	2	(4, 4, 0)	$C_{440}h_0^4\rho^4$
	3	(2, 6, 0)	$C_{260}h_0^2\rho^6$
	4	(0, 8, 0)	$C_{080}\rho^8$
1	0	(7, 1, 1)	$C_{711}h_0^7\rho\cos\phi$
	1	(5, 3, 1)	$C_{531}h_0^5\rho^3\cos\phi$
	2	(3, 5, 1)	$C_{351}h_0^3\rho^5\cos\phi$
	3	(1, 7, 1)	$C_{171}h_0\rho^7\cos\phi$
2	0	(6, 2, 2)	$C_{622}h_0^6\rho^2(\cos\phi)^2$
	1	(4, 4, 2)	$C_{442}h_0^4\rho^4(\cos\phi)^2$
	2	(2, 6, 2)	$C_{262}h_0^2\rho^6(\cos\phi)^2$
	3	(0, 8, 2)	No
3	0	(5, 3, 3)	$C_{533}h_0^5\rho^3(\cos\phi)^3$
	1	(3, 5, 3)	$C_{353}h_0^3\rho^5(\cos\phi)^3$
	2	(1, 7, 3)	$C_{173}h_0\rho^7(\cos\phi)^3$
4	0	(4, 4, 4)	$C_{444}h_0^4\rho^4(\cos\phi)^4$
	1	(2, 6, 4)	$C_{264}h_0^2\rho^6(\cos\phi)^4$
	2	(0, 8, 4)	No
5	0	(3, 5, 5)	$C_{355}h_0^3\rho^5(\cos\phi)^5$
	1	(1, 7, 5)	$C_{175}h_0\rho^7(\cos\phi)^5$
6	0	(2, 6, 6)	$C_{266}h_0^2\rho^6(\cos\phi)^6$
	1	(0, 8, 6)	No
7	0	(1, 7, 7)	$C_{177}h_0\rho^7(\cos\phi)^7$
8	0	(0, 8, 8)	No

The method in this study was further applied to determine all the components of the tertiary aberrations ($q = 4$) (see Table 2). Comparing the results in Table 2 with those in Table 3–3 of [5], it is found that six components (i.e., $C_{173}h_0\rho^7(\cos\phi)^3$, $C_{264}h_0^2\rho^6(\cos\phi)^4$, $C_{355}h_0^3\rho^5(\cos\phi)^5$, $C_{175}h_0\rho^7(\cos\phi)^5$, $C_{266}h_0^2\rho^6(\cos\phi)^6$ and $C_{177}h_0\rho^7(\cos\phi)^7$) are missing from equation (1).

3 Conclusions

The wavefront aberrations $W(h_0, \rho, \phi)$ of axis-symmetrical systems are generally decomposed into their various components using equation (1). However, the numerical results presented in [12] show that this equation cannot guarantee that all of the components of the primary aberrations can be found. In other words, the equation is a sufficient condition only, not a necessary and sufficient condition.

Accordingly, this study has presented a method for determining the possible domains of the non-negative integer indices, m and p, in equation (1) such that all of the components of the aberrations can be found. It has been shown that the index j computed from equation (2) may be negative. Furthermore, three and six new components of the secondary and tertiary aberrations of an axis-symmetrical system have been found, where these components all satisfy the equations describing the fundamental axis-symmetrical nature of axis-symmetrical systems. Overall, the method proposed in this study provides a systematic and robust approach for ensuring that all of the components of any order wavefront aberration in an axis-symmetrical system can be found.

Conflict of interest

The author declares no conflicts of interest.

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